ONLINE SUPPLEMENT FOR FAST AND ACCURATE PROPAGATION OF COHERENT LIGHT

RYAN D. LEWIS, GREGORY BEYLKIN, AND LUCAS MONZÓN

Appendix A. Accurate Propagation in the Fourier Domain

We observed in §2(a) that if the propagation distance z is small, then the propagation problem may be accurately and efficiently solved in the Fourier domain. This is not a new observation—see, e.g., [16, 14]—so our objective here is simply to demonstrate how this propagation method (the so-called angular spectrum method) can be implemented using the quadratures from Theorem 2 to ensure any user-specified accuracy.

Given boundary data f and a propagation distance z, the angular spectrum method amounts to the numerical evaluation of the integral

$$u(\mathbf{x}) = \int\limits_{\|\mathbf{p}\| \le c} \widehat{f}(\mathbf{p})\, \widehat{K}_z(\|\mathbf{p}\|)\, e^{i2\pi \mathbf{x}\cdot\mathbf{p}}\, d\mathbf{p}, \tag{A.1}$$

where $\widehat{K}_z$ is given in (2.5) and

$$\widehat{f}(\mathbf{p}) = \int\limits_{A} f(\mathbf{y})\, e^{-i2\pi \mathbf{y}\cdot\mathbf{p}}\, d\mathbf{y} \tag{A.2}$$

is the Fourier transform of the boundary data. In (A.1) and (A.2) the boundary data has already been replaced by its space-limited and band-limited version, as described in §2(d), so that the function f is concentrated in a square aperture $A = \left[-\frac{a}{2}, \frac{a}{2}\right]^2$ and is band-limited to a disk of radius c for some user-specified accuracy ϵ.

The numerical evaluation of (A.1) amounts to solving a quadrature problem. That is, for accuracy ϵ, we seek quadrature nodes $\mathbf{p}_\ell$, $\ell = 1, \ldots, L$, and associated weights ω_ℓ such that for all $\mathbf{x} \in \left[-\frac{w}{2}, \frac{w}{2}\right]^2$,

$$\left| \int\limits_{\|\mathbf{p}\| \le c} \widehat{f}(\mathbf{p})\, \widehat{K}_z(\|\mathbf{p}\|)\, e^{i2\pi \mathbf{x}\cdot\mathbf{p}}\, d\mathbf{p} - \sum_{\ell=1}^{L} \omega_\ell \widehat{f}(\mathbf{p}_\ell)\, \widehat{K}_z(\|\mathbf{p}_\ell\|)\, e^{i2\pi \mathbf{x}\cdot\mathbf{p}_\ell} \right| \le \frac{\epsilon\, \|f\|_1}{z}.$$

The construction of such quadrature rules for two-dimensional integrals of space-limited and band-limited functions, computed over disks and accurate for any user-specified accuracy ϵ, was described in [6]. The resulting near-optimal quadrature

The authors are with the Department of Applied Mathematics, University of Colorado at Boulder, Boulder, CO 80309, USA. E-mail: lewisrd@colorado.edu; beylkin@colorado.edu (corresponding author); lucas.monzon@colorado.edu. This research was partially supported by NSF grants DMS-1009951, DGE-0801680, DMS-0602284, and DOE/ORNL grant 4000038129.

nodes lie on a polar grid, so that

$$\mathbf{p}_\ell = \mathbf{p}_{mm'} = \rho_m \left(\cos\phi_{mm'}, \sin\phi_{mm'}\right)$$

for some radial nodes ρ_m, $m = 1, \ldots, M$, and angular nodes $\phi_{mm'}$, $m = 1, \ldots, M$ and $m' = 1, \ldots, M'_m$. The total number of nodes $L = L\left(c', \epsilon\right)$ is nearly optimal and depends only weakly on the accuracy ϵ and quadratically on the bandlimit c' of the integrand. We estimate c' as $c' = c\left(a + w\right)/\sqrt{2} + c_K$, where c_K is the bandlimit of $\widehat{K}_z$ in the domain of integration. Estimating the shortest period of oscillation of $\widehat{K}_z$ in the disk of radius c yields

$$c_K \approx \begin{cases} \frac{z}{\left(c^2 z^2 + 2z\sqrt{1-c^2} - 1\right)^{\frac{1}{2}} - cz}, & \text{if } c < 1 \\ z^2 + z\sqrt{z^2 - 1}, & \text{if } c \geq 1. \end{cases} \tag{A.3}$$

Once we specify the location of the desired output samples $\mathbf{x}_n \in \left[-\frac{w}{2}, \frac{w}{2}\right]^2$, $n = 1, \ldots, N$, we compute u rapidly by evaluating

$$\widetilde{u}\left(\mathbf{x}_n, z\right) = \sum_{\ell=1}^{L} \omega_\ell \widehat{f}\left(\mathbf{p}_\ell\right) \widehat{K}_z\left(\|\mathbf{p}_\ell\|\right) e^{i2\pi \mathbf{x}_n \cdot \mathbf{p}_\ell}, \tag{A.4}$$

which requires a single USFFT.

The evaluation of (A.4) requires values of $\widehat{f}$ at the quadrature nodes $\mathbf{p}_\ell$. In most cases, $\widehat{f}$ is not known explicitly and must be computed by numerically evaluating (A.2). This is exactly the quadrature problem addressed in Theorem 2, and we obtain a tensor product grid of quadrature nodes $\mathbf{y}_{jj'} = (y_j, y_{j'})$, $j, j' = 1, \ldots, J$, and associated weights $\tau_j \tau_{j'}$ such that for each $\mathbf{p}_\ell$, $\ell = 1, \ldots, L$,

$$\left| \int_A f\left(\mathbf{y}\right) e^{-i2\pi \mathbf{y} \cdot \mathbf{p}_\ell} \, d\mathbf{y} - \sum_{j,j'=1}^{J} \tau_j \tau_{j'} f\left(\mathbf{y}_{jj'}\right) e^{-i2\pi \mathbf{y}_{jj'} \cdot \mathbf{p}_\ell} \right| \leq \epsilon \left\|f\right\|_1 . \tag{A.5}$$

Thus, $\widehat{f}\left(\mathbf{p}_\ell\right)$ may be computed accurately and rapidly with one USFFT.

Taken together, formulae (A.4) and (A.5) allow us to evaluate the field with any user-specified accuracy with only two USFFTs. This method is simple and effective, provided that the bandlimit c_K in (A.3), which increases with the distance z, is moderate. If c_K is so large that the number of quadrature nodes required to compute (A.4) accurately makes its evaluation infeasible, then the propagation problem should be treated in the spatial domain, using the method we describe in §3.

Remark 1. The dependence of the bandlimit c_K in (A.3) on z is nearly linear provided that $c < 1$, but it depends almost quadratically on z if $c \geq 1$. In the latter case, evanescent waves are present in the solution, and very fine sampling is required to propagate them accurately. We note that in virtually all cases of practical interest, the contribution of evanescent waves may be neglected entirely after only a few wavelengths, so in practice, we may assume that $c < 1$ unless z is small, and therefore the bandlimit $c_K = z^2 + z\sqrt{z^2 - 1}$ is not impractically large.

Appendix B. A Comment on the Fraunhofer Approximation

If the propagation distance z is large with respect to the sizes of the input aperture and the output window, it is common to estimate the field $u\left(\mathbf{x}, z\right)$ using the Fraunhofer approximation (2.13). We note that many optics texts derive this

far-field approximation by approximating the Fresnel approximation. This is unfortunate, because the result is only valid near the optical axis and, therefore, the size of the output region where the asymptotics are accurate is severely restricted. In contrast, there is a well known asymptotic approximation of solutions to the Helmholtz equation that is valid in a much larger output region, and which may be evaluated with a single USFFT. For completeness, we now recall this alternative far-field approximation and relate it to the Fraunhofer approximation.

Recall Lord Rayleigh's integral formula for $u(\mathbf{x}, z)$,

$$u(\mathbf{x}, z) = -\frac{1}{2\pi} \int_A f(\mathbf{y}) \frac{\partial}{\partial z} \left(\frac{e^{i2\pi R}}{R} \right) d\mathbf{y}, \tag{B.1}$$

where A is the input aperture, $f(\mathbf{y}) = u(\mathbf{y}, 0)$ is the boundary data, and $R = \sqrt{z^2 + \|\mathbf{x} - \mathbf{y}\|^2}$. Performing the indicated differentiation gives

$$\frac{\partial}{\partial z} \left(\frac{e^{i2\pi R}}{R} \right) = z e^{i2\pi R} \left(\frac{i2\pi}{R^2} - \frac{1}{R^3} \right),$$

which has the large-R asymptotic approximation

$$\frac{\partial}{\partial z} \left(\frac{e^{i2\pi R}}{R} \right) \sim \frac{i2\pi z e^{i2\pi R}}{R^2}, \quad R \to \infty. \tag{B.2}$$

We now write R as

$$R = \sqrt{z^2 + \|\mathbf{x}\|^2} \left(1 + \frac{\|\mathbf{y}\|^2}{z^2 + \|\mathbf{x}\|^2} - 2\frac{\mathbf{x} \cdot \mathbf{y}}{z^2 + \|\mathbf{x}\|^2} \right)^{\frac{1}{2}}$$

and observe that, since $\mathbf{y} \in A$ and z is much larger than the size of the aperture A, $z \gg \|\mathbf{y}\|$. Hence, we obtain the asymptotic approximation

$$\begin{aligned} R &\sim \sqrt{z^2 + \|\mathbf{x}\|^2} \left(1 - 2\frac{\mathbf{x} \cdot \mathbf{y}}{z^2 + \|\mathbf{x}\|^2} \right)^{\frac{1}{2}} \\ &\sim \sqrt{z^2 + \|\mathbf{x}\|^2} \left(1 - \frac{\mathbf{x} \cdot \mathbf{y}}{z^2 + \|\mathbf{x}\|^2} \right), \quad \mathbf{y} \in A,\ z \to \infty. \end{aligned} \tag{B.3}$$

We emphasize that we did not need to assume that $\|\mathbf{x}\|$ is small, i.e., close to the optical axis, to obtain this approximation. Following the standard procedure, we substitute (B.3) into the exponent in (B.2) and $R \sim \sqrt{z^2 + \|\mathbf{x}\|^2}$ into the denominator, and then use the resulting asymptotic Green's function in (B.1) to arrive at

$$u(\mathbf{x}, z) \sim -\frac{iz e^{i2\pi\sqrt{z^2 + \|\mathbf{x}\|^2}}}{z^2 + \|\mathbf{x}\|^2} \widehat{f} \left(\frac{\mathbf{x}}{\sqrt{z^2 + \|\mathbf{x}\|^2}} \right), \quad z \to \infty. \tag{B.4}$$

The result in (B.4) relates the far-field diffraction pattern to the scaled Fourier transform of the boundary data. We accurately and rapidly evaluate this formula using generalized Gaussian quadratures for band-limited functions from Theorem 2 and the USFFT, in exactly the same manner as in the evaluation of (A.5).

To obtain the Fraunhofer approximation, we could assume that $\|\mathbf{x}\| \ll z$ and make a further approximation by retaining one and two terms of the expansion

$\sqrt{z^2+\|\mathbf{x}\|^2} = z\left(1+\frac{1}{2}\frac{\|\mathbf{x}\|^2}{z^2}+\cdots\right)$ in the denominators and exponent of (B.4), respectively, yielding the standard Fraunhofer approximation (cf. [12, §4.3]),

$$u(\mathbf{x},z) \approx \frac{e^{2\pi i z} e^{i\frac{\pi}{z}\|\mathbf{x}\|^2}}{iz} \widehat{f}\left(\frac{\mathbf{x}}{z}\right). \tag{B.5}$$

However, this additional approximation is not advisable since, unlike (B.4), the Fraunhofer approximation is only valid for points close to the the optical axis, i.e., $\|\mathbf{x}\| \ll z$. There is no computational advantage to be gained by using (B.5), with its restricted region of validity, instead of (B.4), valid for all $\mathbf{x}$ provided that z is sufficiently large, since both of these formulae may be evaluated at the same cost with a single USFFT.

Appendix C. Supplementary information and estimates

C.1. **Algorithm for Approximation by Exponential Sums.** We approximate, for any user-specified accuracy ϵ, a smooth function $f(x)$, $0 \le x \le 1$, by a linear combination of exponentials,

$$\left| f(x) - \sum_{\ell=1}^{L} w_\ell e^{-\eta_\ell x} \right| \le \epsilon, \quad x \in [0,1], \tag{C.1}$$

where the number of complex-valued weights w_ℓ and exponents η_ℓ is nearly minimal. We obtain this representation by solving a discrete version of the approximation problem. Given $2N+1$ evenly-spaced samples of $f(x)$ and target accuracy $\epsilon > 0$, we find the (nearly) minimal number of complex-valued weights w_ℓ and nodes γ_ℓ such that

$$\left| f\left(\frac{k}{2N}\right) - \sum_{\ell=1}^{L} w_\ell \gamma_\ell^k \right| \le \epsilon, \quad 0 \le k \le 2N. \tag{C.2}$$

We must choose the number of samples $2N+1$ large enough so that the function can be accurately reconstructed from its samples. As a result, we obtain the solution to the continuous problem (C.1) from the solution to the discrete problem (C.2) by setting $\eta_\ell = -2N \log \gamma_\ell$. We now describe the algorithm given in [8] (see also [9]) to obtain approximation (C.2).

- Build the $N+1 \times N+1$ Hankel matrix
$$\mathbf{H}_{jk} = f\left(\frac{j+k}{2N}\right), \quad j,k \in [0,N].$$
- Find a vector $\mathbf{u} = (u_0,\ldots,u_N)^T$ satisfying
$$\mathbf{H}\mathbf{u} = \sigma\overline{\mathbf{u}},$$
with positive σ close to the target accuracy ϵ. A problem of this form is known as a con-eigenvalue problem (see, e.g., [13, §4.6]), $\mathbf{u}$ is a con-eigenvector, and σ is a con-eigenvalue. In our case, $\mathbf{H}$ is a Hankel matrix and hence symmetric; the existence of a solution $(\sigma,\mathbf{u})$ follows from Takagi's factorization (see, e.g., [8, pp. 22]), as does the fact that we may take σ to be a singular value of $\mathbf{H}$ and $\mathbf{u}$ to be a specific singular vector.
- Given singular values $\sigma_0 \ge \sigma_1 \ge \ldots \ge \sigma_N$, we select a sufficiently small σ_L, which determines the accuracy of approximation, and the corresponding singular vector $\mathbf{u} = (u_0,\ldots,u_N)^T$.

- Compute the roots γ_ℓ of the con-eigenpolynomial $u(z) = \sum_{n=0}^{N} u_n z^n$ whose coefficients are the entries of the vector $\mathbf{u}$ from the previous step.
- Obtain the weights w_ℓ by solving the least-squares Vandermonde system

$$\sum_{\ell=1}^{N} w_\ell \gamma_\ell^k = f\left(\frac{k}{2N}\right), \quad 0 \le k \le 2N. \tag{C.3}$$

  Typically, only L weights w_ℓ have absolute value larger than the target accuracy ϵ. We then retain only those nodes γ_ℓ that correspond to the significant weights and solve the corresponding Vandermonde system (C.3) again.

The theory underlying this algorithm may be found in [8] and may be traced back to the work of Adamjan, Arov, and Krein [2, 3, 4].

C.2. **Proof of Lemma 3.1.**

Proof. Recall that the SVDs of the $N \times M$ matrices $\mathbf{S}^{(\ell)}$, $\ell = 1, \ldots, L$, are given by

$$\mathbf{S}^{(\ell)}_{nm} = \sum_{q=1}^{\min(M,N)} \sigma^{(\ell)}_q \mathbf{U}^{(\ell)}_{nq} \mathbf{V}^{(\ell)}_{mq}. \tag{C.4}$$

Using (C.4) we have

$$\begin{aligned}
&\mathbf{S}^{(\ell)}_{nm}\mathbf{S}^{(\ell)}_{n'm'} - \sum_{q=1}^{I^{(\ell)}} \sigma^{(\ell)}_q \mathbf{U}^{(\ell)}_{nq} \mathbf{V}^{(\ell)}_{mq} \sum_{s=1}^{I^{(\ell)}} \sigma^{(\ell)}_s \mathbf{U}^{(\ell)}_{n's} \mathbf{V}^{(\ell)}_{m's} \\
&\quad = \mathbf{S}^{(\ell)}_{nm}\mathbf{S}^{(\ell)}_{n'm'} \\
&\qquad - \left(\mathbf{S}^{(\ell)}_{nm} - \sum_{q=I^{(\ell)}+1}^{\min(M,N)} \sigma^{(\ell)}_q \mathbf{U}^{(\ell)}_{nq} \mathbf{V}^{(\ell)}_{mq}\right)\left(\mathbf{S}^{(\ell)}_{n'm'} - \sum_{s=I^{(\ell)}+1}^{\min(M,N)} \sigma^{(\ell)}_s \mathbf{U}^{(\ell)}_{n's} \mathbf{V}^{(\ell)}_{m's}\right) \\
&\quad = \mathbf{S}^{(\ell)}_{nm}\left(\sum_{s=I^{(\ell)}+1}^{\min(M,N)} \sigma^{(\ell)}_s \mathbf{U}^{(\ell)}_{n's} \mathbf{V}^{(\ell)}_{m's}\right) + \mathbf{S}^{(\ell)}_{n'm'}\left(\sum_{q=I^{(\ell)}+1}^{\min(M,N)} \sigma^{(\ell)}_q \mathbf{U}^{(\ell)}_{nq} \mathbf{V}^{(\ell)}_{mq}\right) \\
&\qquad - \left(\sum_{q=I^{(\ell)}+1}^{\min(M,N)} \sigma^{(\ell)}_q \mathbf{U}^{(\ell)}_{nq} \mathbf{V}^{(\ell)}_{mq}\right)\left(\sum_{s=I^{(\ell)}+1}^{\min(M,N)} \sigma^{(\ell)}_s \mathbf{U}^{(\ell)}_{n's} \mathbf{V}^{(\ell)}_{m's}\right).
\end{aligned}$$

Observing that

$$\left|\mathbf{U}^{(\ell)}_{nq}\right|, \left|\mathbf{V}^{(\ell)}_{mq}\right| \le 1 \quad \text{and} \quad \left|\mathbf{S}^{(\ell)}_{nm}\right|, \left|\mathbf{S}^{(\ell)}_{n'm'}\right| \le e^{\frac{\left|\alpha^{(\ell)}\right|}{2} aw},$$

it follows that

$$\left|\mathbf{S}^{(\ell)}_{nm}\mathbf{S}^{(\ell)}_{n'm'} - \sum_{q,s=1}^{I^{(\ell)}} \sigma^{(\ell)}_q \sigma^{(\ell)}_s \mathbf{U}^{(\ell)}_{nq} \mathbf{U}^{(\ell)}_{n's} \mathbf{V}^{(\ell)}_{mq} \mathbf{V}^{(\ell)}_{m's}\right| \le \epsilon_R 2 e^{\frac{\left|\alpha^{(\ell)}\right|}{2} aw},$$

where we have neglected the term

$$\left(\sum_{q=I^{(\ell)}+1}^{\min(M,N)} \sigma^{(\ell)}_q \left|\mathbf{U}^{(\ell)}_{nq} \mathbf{V}^{(\ell)}_{mq}\right|\right)\left(\sum_{s=I^{(\ell)}+1}^{\min(J,M)} \sigma^{(\ell)}_s \left|\mathbf{U}^{(\ell)}_{n's} \mathbf{V}^{(\ell)}_{m's}\right|\right)$$

which has size $\mathcal{O}\left(\epsilon_R^2\right)$. □

C.3. **Rigorous Estimates Relating to Computational Complexity.** The key step in the algorithm described in §3 is the construction of approximation (3.5), where, for a fixed distance z and desired accuracy ϵ_K, we approximate the Rayleigh-Sommerfeld kernel $K_z(r)$ as a linear combination of Gaussians with complex exponents. This approximation must be valid on an interval $0 \leq r \leq r_{\max}$, where, in the case of a square aperture of side-length a and square output window of side-length w, $r_{\max} = (a+w)/\sqrt{2}$. As described in §3, we obtain this approximation by removing the most-oscillatory factor $e^{i\frac{\pi}{z}r^2}$ from $K_z(r)$ then approximating the remaining function, viz. $A_z(r)$ defined in (3.3), using Gaussians with complex exponents,

$$\left| A_z(r) - \sum_{\ell=1}^{L} w_\ell e^{-\eta_\ell r^2} \right| \leq \epsilon_K, \quad 0 \leq r \leq r_{\max}.$$

Three components of this approximation ultimately determine the computational cost of our algorithm:

(1) The number of terms L.
(2) The number of input samples M^2 required to evaluate the integrals in (3.8). This depends on the maximum bandlimit of the integrands, which in turn is determined by the bandlimit of the input function and the values of the exponents η_ℓ.
(3) For each $\ell = 1, \ldots, L$, the number of terms $R^{(\ell)}$ needed in (3.14) to approximate the tensors $\mathbf{T}^{(\ell)}$ defined in (3.10).

We now provide estimates of these quantities.

C.3.1. *Number of Terms Needed to Approximate the Kernel.* It turns out that, for the values of r of interest here, the behavior of the function $A_z(r)$ closely resembles that of $e^{-i\frac{\pi r^4}{4z^3}}$, i.e.,

$$A_z(r) \approx e^{-i\frac{\pi r^4}{4z^3}}, \tag{C.5}$$

which comes from the Taylor series

$$\sqrt{1+\left(\frac{r}{z}\right)^2} = 1 + \frac{1}{2}\left(\frac{r}{z}\right)^2 - \frac{1}{8}\left(\frac{r}{z}\right)^4 + \mathcal{O}\left(\left(\frac{r}{z}\right)^6\right).$$

Recall that one of our goals is to construct an algorithm whose computational cost does not increase with z. Approximation (C.5) implies that the number of terms L needed to approximate A_z will depend on the ratio $\gamma = r_{\max}/z^{\frac{3}{4}}$ (and also on the desired accuracy ϵ_K). We can estimate the number of terms required using techniques similar to those in [9], where functions are approximated as linear combinations of complex Gaussians by manipulating their integral representations. The derivations are somewhat technical, so here we simply present the bound

$$L = L(\gamma, \epsilon) \leq \frac{2\log \epsilon_K^{-1}}{\pi} \times \frac{-B - \sqrt{B^2 - 8B\left(\log \epsilon_K + \log 2\sqrt{\pi B}\right)}}{B - \sqrt{B^2 - 8B\log \epsilon_K}}, \tag{C.6}$$

where

$$B = \frac{\pi\gamma^4}{4}. \tag{C.7}$$

The rightmost factor in (C.6) depends approximately linearly on B and only weakly on ϵ_K, so that $L = \mathcal{O}\left(\gamma^4 \log \epsilon_K^{-1}\right)$. Since the number of terms grows rapidly with γ, we require that $\gamma \leq 2.62$ to ensure that the approximation is efficient. This implies that the maximum output window is given by (4.1), and we assume that a, w, and z satisfy $\gamma \leq 2.62$ for the remainder of this section.

The same integral-based techniques that lead to (C.6) also yield the bounds

$$|\alpha_\ell| \leq \frac{B+D}{r_{\max}^2} \tag{C.8}$$

and

$$|\beta_\ell| \leq \frac{\pi}{z} + \frac{D}{r_{\max}^2}, \tag{C.9}$$

where

$$D = D\left(B, \epsilon_K\right) = \sqrt{B^2 - 8B\left(\log \epsilon_K + \log 2\sqrt{\pi B}\right)}.$$

(Recall from §3(b) that $\alpha_\ell = \mathcal{R}\mathrm{e}\,\eta_\ell$ and $\beta_\ell = \mathcal{I}\mathrm{m}\,\eta_\ell - \frac{\pi}{z}$, where η_ℓ, $\ell = 1, \ldots, L$, are the exponents used to approximate the Rayleigh-Sommerfeld kernel.) We will use these bounds below to determine the number of required input samples M^2 and the number of terms $R^{(\ell)}$ in the approximations (3.14). To simplify the computations that follow, let us estimate their values. Since $\gamma \leq 2.62$, we have from (C.7) that $B \leq 37$. For $\epsilon_K = 10^{-3}$ we have $D \leq 50.1$, and for $\epsilon_K = 10^{-6}$ we have $D \leq 67.5$. Since $r_{\max} = (a+w)/\sqrt{2}$ is typically several thousand wavelengths, and often much larger, we can see that

$$|\alpha_\ell| \ll aw \quad \text{and} \quad |\beta_\ell| \approx \frac{\pi}{z}.$$

C.3.2. *Number of Input Samples.* The number of quadrature nodes (input samples) M^2 required in Theorem 2 to evaluate the integrals in (3.8) is determined by the bandlimits of the integrands and the desired accuracy ϵ_Q. We use the bound (C.9) to estimate the number of input samples required to accurately evaluate the integrals.

We start by rescaling the variables $\mathbf{x}$ and $\mathbf{y}$ to the unit square by defining $\mathbf{x} = \frac{w}{2}\mathbf{x}'$ and $\mathbf{y} = \frac{a}{2}\mathbf{y}'$, so that the integrals in (3.8) become

$$\frac{a^2}{4} \int\limits_{[-1,1]^2} f\left(\frac{a}{2}\mathbf{y}'\right) e^{-\frac{(\alpha_\ell + i\beta_\ell)a^2}{4}\|\mathbf{y}'\|^2} e^{\frac{\alpha_\ell a w}{4}\mathbf{x}'\cdot\mathbf{y}'} e^{i\frac{\beta_\ell a w}{2}\mathbf{x}'\cdot\mathbf{y}'}\, d\mathbf{y}', \quad \ell = 1, \ldots, L.$$

The bandlimit of the integrand is the sum of the bandlimits of each of the factors. Let c be the bandlimit of the the (rescaled) input function $f\left(\frac{a}{2}\mathbf{y}'\right)$. We now estimate the bandlimits of the other factors in the integrands.

- When estimating the bandlimit of $e^{-\frac{(\alpha_\ell + i\beta_\ell)a^2}{4}\|\mathbf{y}'\|^2}$, we may neglect the influence of α_ℓ since $|\alpha_\ell| \ll a^2$. We may also use the approximation $|\beta_\ell| \approx \frac{\pi}{z}$. Since, in the unit square, the shortest period of oscillation of the function $e^{-i\frac{\pi a^2}{4z}\|\mathbf{y}'\|^2}$ is $\sqrt{2}\left(\sqrt{1+\frac{4z}{a^2}} - 1\right) \approx \frac{2\sqrt{2}z}{a^2}$, we estimate the bandlimit of this term as $c_2 = \frac{a^2}{2\sqrt{2}z}$.
- Since $|\alpha_\ell| \ll aw$, the factor $e^{\frac{\alpha_\ell a w}{4}\mathbf{x}'\cdot\mathbf{y}'}$ does not significantly impact the bandlimit of the integrand, so we neglect it completely.

- Because $\|\mathbf{x}'\| \leq \sqrt{2}$, the bandlimit of the factor $e^{i\frac{\beta_\ell aw}{2}\mathbf{x}'\cdot\mathbf{y}'}$ is $c_3 = \frac{\pi aw}{\sqrt{2}z}$, where we used the approximation $|\beta_\ell| \approx \frac{\pi}{z}$.

Thus, the bandlimit of the integrand is approximately

$$c' = c + c_2 + c_3 = c + \frac{a^2}{2\sqrt{2}z} + \frac{\pi aw}{\sqrt{2}z}. \tag{C.10}$$

From [7], we have that, for a desired accuracy ϵ_Q, the number of samples required to evaluate the integrals (3.8) satisfies $M^2 = \mathcal{O}\left((c')^2 \log^2 \epsilon_Q^{-1}\right)$.

C.3.3. *Number of Terms Needed to Approximate the Tensors.* Now let us use the bound (C.8) to estimate the number of terms required in the approximations (3.14). The tensors $\mathbf{S}^{(\ell)}$ in (3.11) are discrete approximations of the functions

$$S^{(\ell)}(x,y) = e^{2\alpha_\ell xy}, \quad x \in \left[-\frac{w}{2}, \frac{w}{2}\right],\ y \in \left[-\frac{a}{2}, \frac{a}{2}\right], \text{ and } \ell = 1, \ldots, L,$$

which have the Chebyshev expansions

$$e^{2\alpha_\ell xy} = J_0(-i\alpha_\ell ax) + 2\sum_{n=1}^{\infty} i^n J_n(-i\alpha_\ell ax)\, T_n\left(\frac{2y}{a}\right),$$

where J_n is the n-th order Bessel function of the first kind and T_n is the n-th degree Chebyshev polynomial of the first kind. For fixed x, the magnitude of the Bessel functions decay super-exponentially as $n \to \infty$. In fact, using [1, (9.1.62)], we have the bound

$$|J_n(-i\alpha_\ell ax)| \leq \frac{|\alpha_\ell ax|^n e^{|\alpha_\ell ax|}}{2^n n!}.$$

Now observe that $|\alpha_\ell ax| \leq D\frac{aw}{(a+w)^2}$, where we used (C.8) and the fact that $r_{\max} = (a+w)/\sqrt{2}$. For some desired accuracy ϵ_R, let P be the smallest integer such that $\left|J_n\left(-iD\frac{aw}{(a+w)^2}\right)\right|^2 \leq \epsilon_R$ for all $n \geq P$. Then we may estimate the number of terms $R^{(\ell)}$ in (3.14) as

$$R^{(\ell)} \leq (P+1)^2 = \mathcal{O}\left(\log \epsilon_R^{-1}\right).$$

If we assume that the output window is at least as large as the input aperture, i.e., $w \geq a$, then the argument of the Bessel function satisfies

$$\left|D\frac{aw}{(a+w)^2}\right| \leq \frac{D}{4},$$

and it is easy to verify that the numerical rank R of each matrix $\mathbf{S}^{(\ell)}$ satisfies $R \leq 19$ for $\epsilon_R = 10^{-3}$ and $R \leq 28$ for $\epsilon_R = 10^{-6}$.

C.4. **A Comment on the Fresnel Approximation and the Rayleigh-Sommerfeld Integral.** The limited accuracy of the Fresnel approximation for propagating boundary data is well known and there have been many publications to estimate the error and correct for it (see, e.g., [15, 17, 10] and references therein). However, the resulting corrections to the Fresnel approximation fail to provide a numerically stable procedure to control the error within a user-specified accuracy. For example, [11] expresses the field as a series where the first term is the estimate of the field obtained via the Fresnel approximation and where the higher-order terms may be viewed as correction terms. However, applying these correction terms requires the evaluation of high-order derivatives of the Fresnel approximation (which is itself obtained from

experimentally-determined data). Such a scheme does not provide a mechanism to control error, i.e., the user cannot specify any desired accuracy $\epsilon > 0$ in advance and be assured that the level of error in the computed solution is bounded by ϵ. We note that improvements to the Fresnel approximation can be obtained for special types of boundary data—for example [5] describes such a scheme for the special case of a converging spherical wave. However, we are not aware of any numerical method based on the Fresnel approximation that can propagate arbitrary boundary data with controlled error.

Although we state that our method can be viewed as a generalization of the Fresnel approximation—indeed, it can be— it is, in fact, not directly related to this approximation (including the alternative form discussed in, e.g., [10]). Our results provide an algorithm to accurately compute the Rayleigh-Sommerfeld integral with any user-specified accuracy and it just happens that the form of approximation resembles the Fresnel approximation. We think it is worthwhile to point out this resemblance but, in essence, our paper does not deal with the Fresnel approximation as such.

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