BIT 9 (1969), 78–80

SCIENTIFIC NOTES

A METHOD FOR ACCELERATION OF THE CONVERGENCE OF INFINITE SERIES

ANDERS BECKMAN, BENGT FORNBERG and ARNE TENGVALD

A frequently encountered problem is the calculation of the sum of a convergent positive infinite series. We will here present a method for the approximation of such sums. The method originates from an idea by Salzer, where a number of partial sums S_n are calculated,

$$S_n = \sum_{v=1}^{n} f(v) .$$

These are then regarded as functions of $g(n) = 1/n$, and polynomial extrapolation is made to $g(n) = 0$.

For this extrapolation to give acceptable results, S_n considered as a function of $g(n)$ must behave approximately like a polynomial when $g(n)$ approaches zero, that is, the slope of its tangent must not locally approach either zero or infinity. Consequently, the limit

$$(1) \qquad \lim_{n\to\infty} \frac{S_\infty - S_n}{g(\infty) - g(n)} \approx \lim_{n\to\infty} \frac{\int_n^\infty f(t)\,dt}{-g(n)} = \lim_{n\to\infty} \frac{f(n)}{g'(n)} = \varphi$$

must exist and be different from zero. Thus we should choose $g(n)$ such that $g'(n) = k \cdot f(n)$, $k \approx$constant, and $g(\infty) = 0$. We now easily see when the choice $g(n) = 1/n$ works, namely when the terms of the series decrease as $1/n^2$. However, if for other series we choose $g(n)$ according to the formula above, we get a very pronounced increase in effectiveness.

A very useful method for polynomial extrapolations of this kind is Aitken's method for iterated inter/extrapolation since it yields successive results which can be used to estimate the error of the extrapolation. A further advantage is that it can easily be programmed.

We now exemplify the effectiveness of a correct choice of $g(n)$. The table below also exemplifies the interesting fact that an incorrect choice of $g(n)$ yields a monotonic series of extrapolation results while the errors vary in sign irregulary for the correct choice of $g(n)$. This

Received Oct. 26, 1968.

Table 1.

I	II	III	IV	V
4	1.67100348	1.6710034803	$9.414 \cdot 10^{-1}$	$9.414 \cdot 10^{-1}$
8	1.92667668	2.5439263851	$6.845 \cdot 10^{-2}$	$4.300 \cdot 10^{-1}$
12	2.04680288	2.6323607805	$-1.999 \cdot 10^{-2}$	$2.730 \cdot 10^{-1}$
16	2.12006585	2.6145218343	$-2.146 \cdot 10^{-3}$	$2.020 \cdot 10^{-1}$
20	2.17068207	2.6120597791	$3.156 \cdot 10^{-4}$	$1.608 \cdot 10^{-1}$
24	2.20833536	2.6123594597	$1.589 \cdot 10^{-5}$	$1.336 \cdot 10^{-1}$
28	2.23775543	2.6123793502	$-4.002 \cdot 10^{-6}$	$1.144 \cdot 10^{-1}$
32	2.26156252	2.6123748299	$5.188 \cdot 10^{-7}$	$1.000 \cdot 10^{-1}$
36	2.28134076	2.6123753354	$1.320 \cdot 10^{-8}$	$8.883 \cdot 10^{-2}$
40	2.29811165	2.6123753631	$-1.449 \cdot 10^{-8}$	$7.992 \cdot 10^{-2}$

I: Number of terms in the partial sum S_m.
II: $S_m = \sum_{v=1}^{m} v^{-1.5}$.
III: Extrapolated results from polynomial of degree $m/4-1$, using the $m/4$ first listed partial sums; $g(n) = n^{-0.5}$.
IV: Error with $g(n) = n^{-0.5}$. The correct value of the sum is $2.612375348685\underline{5}$.
V: Error with the incorrect choice $g(n) = n^{-1}$.

enables us to find the correct $g(n)$ even for series with terms which are not explicitly defined.

If we choose

$$g(n) = \int_{n+\frac{1}{2}}^{\infty} f(x)dx$$

then (1) gives $\varphi = -1$, and a second degree interpolation polynomial

$$P(g(n)) = S_\infty - g(n) + kg(n)^2$$

should yield a good approximation since

$$P'(g(\infty)) = P'(0) = -1 = \varphi .$$

If we have two partial sums S_a and S_b then

$$S_a = S_\infty - g(a) + kg(a)^2$$
$$S_b = S_\infty - g(b) + kg(b)^2$$

which implies

$$S_\infty = \frac{(S_a + g(a))g(b)^2 - (S_b + g(b))g(a)^2}{g(b)^2 - g(a)^2}$$

This method of extrapolation uses our knowledge of the behavior of the sumfunction for $g(n)$ close to zero and should consequently yield a comparatively better result for slowly convergent series.

We now apply this method to $\sum_{n=1}^{\infty} n^{-1.5}$

$$S_{10} = 1.99534$$
$$S_{20} = 2.17068$$
$$S_{30} = 2.25024$$
$$S_{40} = 2.29811$$

S_{10} and S_{40} yield $S_\infty = 2.61232$
S_{20} and S_{40} yield $S_\infty = 2.61236$
S_{30} and S_{40} yield $S_\infty = 2.61237$

As is readily seen, the methods given above can easily be used to calculate integrals of the type

$$\int_a^\infty f(x)\,dx$$

where $f(x)$ is a positive function. Here $g(n)$ should be an approximation of $\int_n^\infty f(x)dx$ which is asymptotically correct when n tends to infinity. For a fixed n, the approximation need not be very accurate.

DEPARTMENT OF COMPUTER SCIENCES
UNIVERSITY OF UPPSALA
SWEDEN