H

HAMILTONIAN SYSTEMS

A system of $2n$, first order, ordinary differential equations

$$\dot{z} = J\nabla H(z,t), \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{1}$$

is a Hamiltonian system with n degrees of freedom. (When this system is non-autonomous, it has $n + 1/2$ degrees of freedom.) Here H is the Hamiltonian, a smooth scalar function of the extended phase space variables z and time t, the $2n \times 2n$ matrix J is called the "Poisson matrix", and I is the $n \times n$ identity matrix. The equations naturally split into two sets of n equations for *canonically conjugate* variables, $z = (q, p)$, as follows.

$$\dot{q} = \partial H/\partial p, \quad \dot{p} = -\partial H/\partial q \ .$$

Here the n coordinates q represent the configuration variables of the system (positions of the component parts) and their canonically conjugate momenta p represent the impetus associated with movement. These equations generalize Newton's third law: $F = ma = \mathrm{d}p/\mathrm{d}t$, to systems (like particles in magnetic fields, or motion in non-inertial reference frames) where the momentum is not simply mass times velocity. The Hamiltonian usually represents the total energy of the system; thus if $H(q, p)$ does not depend explicitly upon t, then its value is invariant, and Equations (1) are a conservative system. More generally, however, Hamiltonian systems need not be conservative.

William Rowan Hamilton first gave this reformulation of Lagrangian dynamics in 1834 (Hamilton, 1835). However, Hamiltonian dynamics is much more than just a reformulation. It leads, for example, to Poincaré's geometrical insight that gave rise to symplectic geometry, and it provides a compact notation in which the concept of integrability is most naturally expressed and in which perturbation theory can be efficiently carried out. Moreover, nearly integrable Hamiltonian systems exhibit a remarkable stability expressed by the famous results of Kolmogorov–Arnold–Moser (KAM) theory and also Nekhoroshev theory. The Hamiltonian formulation also provides the foundation of both statistical and quantum mechanics.

Importantly, virtually all of the dynamical laws of physics—from the motion of a point particle, to the interaction of complex quantum fields—have a formulation based on Equations 1. For example, frictionless mechanical systems are described by a Hamiltonian $H(q, p) = K(p) + V(q)$, where K is the kinetic energy (which is often quadratic in p), and V is the potential energy. For example, an ideal planar pendulum consists of a point particle of mass m attached to a massless rigid rod of length L whose other end is attached to a frictionless pivot. The most convenient configuration variable for the pendulum is $q = \theta$, the angle of the rod from the vertical. The gravitational potential energy of the system is then $V = -mgL\cos\theta$, and its kinetic energy is $K = p^2/(2mL^2)$, where p is the angular momentum about the pivot. For this case Equations (1) become

$$\dot{\theta} = \frac{\partial H}{\partial p} = \frac{p}{mL^2}, \quad \dot{p} = -\frac{\partial H}{\partial q} = -mgL\sin\theta. \tag{2}$$

The point $(0, 0)$ is a stable (elliptic) equilibrium corresponding to the pendulum hanging down, at rest. The point $(\pm\pi, 0)$ is also an equilibrium but is an unstable (saddle) point. The unstable eigenvector of the saddle is the beginning of the unstable manifold, W^{u}, a trajectory that is backwards asymptotic to the saddle. By energy conservation, the unstable manifold corresponds to a branch of the energy contour

$E = mgL$, which again joins the saddle point (after the pendulum has undergone one full rotation). Thus, this trajectory is forward asymptotic to the saddle as well, that is, it lies on the stable manifold, W^s, of the saddle. Orbits of this kind are called "homoclinic". In this case the homoclinic orbit separates the trajectories oscillating about the elliptic equilibrium from those in which the pendulum undergoes complete rotations, thus it is called a *separatrix*. Orbits near the elliptic equilibrium oscillate with the frequency of the linearized, harmonic oscillator, $\omega = \sqrt{g/L}$. The frequency of oscillation decreases monotonically as the amplitude increases, approaching zero logarithmically at the separatrix. The frequency of rotation of the solution grows again from zero as the energy is further increased.

Canonical Structure

The geometrical structure of Hamiltonian systems arises from the preservation of the loop action, defined by

$$A[\gamma] = \oint_\gamma p \, \mathrm{d}q - H \, \mathrm{d}t, \tag{3}$$

where γ is any closed loop in (q, p, t)-space. A consequence of Equations (1) is that if each point on a loop γ_0 is evolved with the flow to obtain a new loop γ_t, then $A[\gamma_0] = A[\gamma_t]$: the loop action is known as the *Poincaré invariant*.

The Hamiltonian form of Equations (1) is not preserved under an arbitrary coordinate transformation (unlike the Euler–Lagrange equations for a Lagrangian system). A canonical transformation $(q, p) \to (q', p')$ preserves the form of the Equations (1). Canonical transformations can be obtained by requiring that the Poincaré invariant of Equation (3) be the same in the new coordinate system, or locally that

$$\left(p' \, \mathrm{d}q' - H' \, \mathrm{d}t\right) - (p \, \mathrm{d}q - H \, \mathrm{d}t) = \mathrm{d}F$$

is the total differential of a function F. If F is represented as a function of a selection of half of the variables (q, p) and the complementary half of (q', p'), it is a "generating" function for the canonical transformation. For example, a function $F(q, q', t)$ implicitly generates a canonical transformation through

$$p = -\frac{\partial F}{\partial q}, \quad p' = \frac{\partial F}{\partial q'},$$
$$H'(q', p', t) = H(q, p, t) - \frac{\partial F}{\partial t}. \tag{4}$$

In order that this transformation be well defined, the first equation must be inverted to find $q'(q, p)$; this requires that the matrix $\partial^2 F/\partial q \partial q'$ be nonsingular. An autonomous canonical transformation is also called a "symplectic map". Canonical transformations are often employed to simplify the equations of motion. For example, Hamilton's equations are especially simple if the new Hamiltonian is a function of only the momentum variables, $H'(p')$. If we can find a transformation to such a coordinate system then the system is said to be integrable. In general, such transformations do not exist; one of the consequences is chaotic motion.

Integrability

Loosely speaking, a set of differential equations is integrable if it can be explicitly solved for arbitrary initial conditions (Zakharov, 1991). The explicit solution, when inverted, yields the initial conditions as invariant functions along the orbits of a system—the initial conditions are constants of motion. A Hamiltonian $H(q, p)$ is said to be Liouville integrable if it can be transformed to a canonical coordinate system in which it depends only on new momenta. When the energy surfaces are compact and the new momenta are everywhere independent, Arnold showed that it is always possible to choose the momentum variables so that their conjugate configuration variables are periodic angles ranging from 0 to 2π (Arnold, 1989). These coordinates are called "action-angle variables", denoted $(\theta, \mathcal{J})$.

As the Hamiltonian is a function only of the actions, $H(\mathcal{J})$, Equation (1) becomes $\dot{\mathcal{J}} = 0$, and $\dot{\theta} = \Omega(\mathcal{J}) = \partial H/\partial \mathcal{J}$. A system is anharmonic when the frequency vector Ω has a nontrivial dependence on $\mathcal{J}$. Thus for an integrable system, motion occurs on the n-dimensional tori $\mathcal{J} =$ constant. Orbits helically wind around the torus with frequencies Ω that depend upon the torus chosen. When the frequency is nonresonant (there is no integer vector m for which $m \cdot \Omega = 0$), then the motion is dense on the torus.

Any one degree-of-freedom, autonomous Hamiltonian system is locally integrable. A Hamiltonian with more than one degree of freedom, such as pendulum with an oscillating support, is typically not integrable. Systems that are separable into non-interacting parts are integrable, and there are also a number of classical integrable systems with arbitrarily many degrees of freedom. These include the elliptical billiard, the rigid body in free space, the Neumann problem of the motion of a particle on a sphere in a quadratic potential, the Toda lattice, and the Calogero–Moser lattice (Arnold, 1988).

Hamiltonian Chaos

The problem of understanding the motion of a slightly perturbed integrable system originated with the desire to understand the motion of the planets. The Kepler problem corresponding to the gravitational interaction of two spherical bodies is integrable; however, once other effects (such as the mutual forces among planets) are included, there appear to be no general, explicit solutions. Poincaré in particular addressed the question of the stability of the solar system, finally realizing that the convergence of perturbation series for the solutions could not be guaranteed, and discovering the phenomenon of transverse homoclinic intersections that is a harbinger of chaos (Poincaré, 1892).

Consider the problem

$$H(\theta, \mathcal{J}) = H_0(\mathcal{J}) + \varepsilon H_1(\theta, \mathcal{J}) + \cdots,$$

where the perturbation H_1 depends periodically on the angles, and can be expanded in a Fourier series. When $\Omega(\mathcal{J})$ is nonresonant, a formal sequence of canonical transformations can be constructed to find a set of coordinates in which H is independent of the angle. The problem is the occurrence of denominators in the coefficients proportional to resonance conditions $m \cdot \Omega(\mathcal{J})$ for integer vectors m. Even for actions where the frequencies are incommensurate, it is always possible to make $m \cdot \Omega(\mathcal{J})$ arbitrarily small by choosing large enough integers m. Thus, a priori bounds on the convergence fail. This is called the "problem of small denominators".

Chirikov realized that small denominators signal the creation of topologically distinct regions of motion (Chirikov, 1979; MacKay & Meiss, 1987). Near a typical resonance, one can use averaging methods to approximate the motion by an integrable pendulum-like Hamiltonian, effectively discarding all of the terms in H_1 except for those that are commensurate with the resonance, that is, the Fourier modes $H_m(\mathcal{J})$ with $m \cdot \Omega = 0$. Thus, orbits near to a resonance are trapped in an effective potential well. The domain of the trapped motion has the width in action of the corresponding pendulum separatrix; it is typically proportional to the square root of the mth Fourier amplitude of H_1. If we can treat the resonances independently, then each gives rise to a corresponding separatrix. However, as the perturbation amplitude grows, this approximation must break down as it predicts the overlap of neighboring separatrices. In 1959, Chirikov proposed this resonance overlap condition as an estimate of the onset of global chaos. Renormalization theory gives a more precise criterion (*See* **Standard map**).

This picture, together with the fact that rational numbers are dense, leads to the expectation that none of the invariant tori of an integrable system persist when it is perturbed with an arbitrarily small perturbation. Surprisingly, the Fermi–Pasta–Ulam computational experiment in 1955 (Fermi et al., 1965; Weissert, 1997) failed to find this behavior. Indeed, KAM theory, initiated by Andrei Kolmogorov in the 1950s and developed in the 1960s by Vladimir Arnold and Jürgen Moser, proves persistence of most of the invariant tori (de la Llave, 2001; Pöschel, 2001). This holds when the perturbation is small enough, provided that the system satisfies an anharmonicity or nondegeneracy condition, is sufficiently differentiable, and the frequency of the torus is sufficiently irrational. The irrationality condition is that $|m \cdot \Omega| > c/|m|^{\tau}$ for all nonzero integer vectors m and some $c > 0$ and $\tau \geq 1$; this is a "Diophantine condition". Each of these conditions is essential, though some systems (like the solar system for which the frequencies are degenerate) can be reformulated so that KAM theory applies.

Resonant tori and tori whose frequencies are nearly commensurate lie between the Diophantine tori. Generally, these tori are destroyed by a small perturbation, and either form new, secondary tori trapped in a resonance, or are replaced by a zone of chaotic motion that is found in the neighborhood of the stable and unstable manifolds of the resonance. These generically intersect and give rise to a homoclinic tangle or trellis that contains a Smale horseshoe. In the case that the Hamiltonian is analytic, the size of the chaotic region is exponentially small in ε, and thus can be difficult to detect (Gelfreich & Lazutkin, 2001).

For small perturbations of an integrable Hamiltonian, it remains an open problem to show that a nonzero volume of initial conditions behaves chaotically, in the sense that they have positive Lyapunov exponents. Numerical investigations indicate that orbits in the chaotic zones do have positive Lyapunov exponents, and that these domains form a "fat fractal" (a fractal with positive measure). There are also many examples of uniformly hyperbolic dynamics (especially for the case of billiards (Bunimovich, 1989) which can also have properties such as mixing and ergodicity.

The problem of the nonlinear stability of a typical system is also open (*See* **Symplectic maps**). However, N.N. Nekhoroshev showed in 1977 that for an analytic system, the actions drift very little for very long times (at most by an amount that is proportional to a power of ε for times that are exponentially long in ε (Lochak, 1993; Pöschel, 1993)). Thus, while it is possible that a KAM torus is unstable, for most practical purposes, they appear to be stable.

Generalizations

Many partial differential equations (PDEs) also have a Hamiltonian structure. For a PDE with independent variables (x, t), the canonical variables are replaced by fields $(q(x,t), p(x,t))$ and the partial derivatives in (1) by functional or Frechêt derivatives, so that

$$\frac{\partial q}{\partial t} = \frac{\delta H}{\delta p}, \quad \frac{\partial p}{\partial t} = -\frac{\delta H}{\delta q}.$$

The Hamiltonian functional H is the integral of an energy density. For example, the wave equation has the Hamiltonian $H[q,p] = \int \mathrm{d}x \frac{1}{2}\left(p^2 + c^2(\partial_x q)^2\right)$. Other nonlinear wave equations such as the integrable nonlinear Schrödinger, Korteweg–de-Vries, and sine-Gordon equations also have Hamiltonian formulations.

JAMES D. MEISS

See also **Adiabatic invariants; Chaotic dynamics; Constants of motion and conservation laws; Ergodic theory; Euler–Lagrange equations; Fermi–Pasta–Ulam (FPU) oscillator chain; Hénon–Heiles system; Horseshoes and hyperbolicity in dynamical systems; Lyapunov exponents; Melnikov method; Pendulum; Phase space; Poisson brackets; Standard map; Symplectic maps; Toda lattice**

Further Reading

Arnold, V.I. (editor). 1988. *Dynamical Systems III*, New York: Springer

Arnold, V.I. 1989. *Mathematical Methods of Classical Mechanics*, New York: Springer

Bunimovich, L.A. 1989. Dynamical systems of hyperbolic type with singularities. In Sinai, Y., (editor), *Dynamical Systems*, Berlin: Springer, p. 278

Chirikov, B.V. 1979. A universal instability of many–dimensional oscillator systems. *Physics Reports*, 52: 265–379

de la Llave, R. 2001. A tutorial on KAM theory. In *Smooth Ergodic Theory and its Applications (Seattle, WA, 1999)*, Providence, RI: American Mathematical Society, pp. 175–292

Fermi, E., Pasta, J. & Ulam, S. 1965. Studies of nonlinear problems. In *Collected Papers of Enrico Fermi*, vol. 2, edited by E.Segré, Chicago: University of Chicago Press; pp. 977–988

Gelfreich, V.G. & Lazutkin, V.F. 2001. Separatrix splitting: perturbation theory and exponential smallness. *Russian Mathematical Surveys*, 56(3): 499–558

Hamilton, W.R. 1835. On the application to dynamics of a general mathematical method previously applied to optics. *British Association Report, 1834*, pp. 513–518

Lochak, P. 1993. Hamiltonian perturbation theory: Periodic orbits, resonances and intermittancy. *Nonlinearity*, 6: 885–904

MacKay, R.S. & Meiss, J.D. (editors). 1987. *Hamiltonian Dynamical Systems: A Reprint Selection*, London: Adam Hilger

Poincaré, H. 1892. *Les Methodes Nouvelles de la Mechanique Celeste*, Paris: Gauthier–Villars

Pöschel, J. 1993. Nekhoroshev estimates for quasi–convex Hamiltonian systems. *Mathematische Zeitschrift*, 213: 187–216

Pöschel, J. 2001. A lecture on the classical KAM theorem. In *Smooth Ergodic Theory and its Applications* (*Seattle, WA1999*), Providence, RI: American Mathematical Society, pp. 707–732

Weissert, T.P. 1997. *The Genesis of Simulation in Dynamics: Pursuing the Fermi–Pasta–Ulam Problem*, New York: Springer

Zakharov, V.E. (editor). 1991. *What is Integrability?*, Berlin: Springer

Manuscript Queries

Title: Encyclopedia of Non-linear Sciences
Alphabet H: Hamiltonian systems

Page	Query Number	Query
No Queries		