A brief introduction to boundary integral equation techniques

P.G. Martinsson, March 2012

Summary: *We describe a set of methods for computing approximate solutions to linear boundary value problems. The foundation of these methods is a reformulation of the partial differential equation as an integral equation. In many cases, the resulting integral equation can rapidly be solved to very high accuracy, and in an entirely stable manner. The computational speed of these methods compares favorably with other numerical techniques for partial differential equations. The most dramatic speed-up occurs for problems that can be reformulated as integral equations defined on the boundary alone.*

1. Introduction

1.1. The basic idea. The set of methods described in this note are applicable to a wide range of linear boundary value problems. For simplicity, we suppose that the BVP we are interested in solving takes the form

$$A\,u = 0 \qquad \text{on } \Omega, \tag{1}$$

$$u = f \qquad \text{on } \Gamma, \tag{2}$$

where A is a linear elliptic partial differential operator with constant coefficients, where Ω is a domain in $\mathbb{R}^d$ with boundary Γ, and where f is a pre-scribed function on Γ.[1] It is in many cases possible to reformulate a BVP such as (1), (2) as a boundary integral equation of the form

$$v(x) + \int_\Gamma k(x,y)\,v(y)\,dl(y) = g(x), \tag{3}$$

where k is a *kernel function* derived from the fundamental solution associated with the operator A, and g is a data function derived from the boundary data f (a specific example is given in Section 3.1). There typically exist several different BIE formulations of the same BVP; choosing the "right" one is important, *cf.* Remark 0.1.

Upon discretization, the BIE (3) turns into a system of N linear algebraic equations that we write

$$\mathbf{v} + \mathbf{A}\,\mathbf{v} = \mathbf{g}, \tag{4}$$

where $\mathbf{A}$ is a matrix approximating the integral operator in (3). An apparent obstacle to solving (4) rapidly is that the matrix $\mathbf{A}$ is almost always dense. However, this matrix typically has properties that allow the computation of products $\mathbf{w} \mapsto \mathbf{A}\,\mathbf{w}$ cheaply. For instance, there exist Fast Multipole Methods (FMMs) for a wide variety of kernels that perform such matrix-vector multiplications in $O(N)$ arithmetic operations, with a small constant. Then iterative solvers (such as GMRES) can be used to determine an approximate solution of (4) very rapidly.

[1] For BVP such as (1), (2), the equivalent integral equations are defined on the boundary alone. This is the environment in which FIEM has the strongest competitive advantage. However, it can profitably be applied to a wide range of other problems as well, including problems with a body load, see Section 1.3.

REMARK 0.1. *The number of iterations required to achieve an accurate solution of (4) using an iterative solver depends strongly on the choice of boundary integral equation formulation. Leaving the details to Section ?, we simply mention here that it is almost always a good idea to use a BIE of the form (3), in which the integral operator is compact in some function space. An equation of this form is called a second kind Fredholm equation. As shown in Section 3, such formulations can lead to extra-ordinarily rapid convergence (double precision accuracy in the linear solve in 15 iterations or less).*

In some environments, there also exist $O(N)$ numerical techniques that do not rely on iterative solvers, but directly compute a representation of the inverse of the operator in (3).

REMARK 0.2 (Terminology). *In this note, the term "fast" has a precise technical meaning: We say that a numerical method is fast if its execution time scales close to linearly with problem size. To be precise, if it has $O(N(\log N)^r)$ complexity for some real number r. (In practice, $r = 0$ can often be achieved, and we rarely need r larger than 2.) The term "Fast Integral Equation Method" (FIEM) is then taken to refer to a combination of an integral equation formulations and a fast solver. Our usage of the term "fast" is standard in this context, but the acronym FIEM is introduced purely for the purposes of this note.*

1.2. Benefits. When a BVP can be rewritten as a second kind Fredholm equation on the boundary, FIEM has several advantages:

(1) The condition number of the numerical problem is typically similar to the condition number of the actual "physics" of the problem. This is in contrast to discrete systems arising from discretizations of differential equations, which typically have very large condition numbers. (High condition numbers lead to several difficulties; loss of accuracy being one.)

(2) The computational complexity is asymptotically optimal in the sense that the number of floating point operations required scales linearly with the true complexity of the problem. In the example in Section 1.1, it scales linearly with the number of degrees of freedom required to discretized the *boundary* of the domain. The data provided is defined on a set of dimension $d - 1$ so we should not have to discretize a set of dimension d.

(3) The construction of discretization schemes that are both stable and of high order is somewhat easier in the integral equation environment than it is in the differential equation environment.

(4) Handling *exterior* problems where the domain Ω is the region outside some given contour or surface Γ is quite hard when discretizing the differential equation. Typically some artificial external boundary is introduced. In contrast, in the FIEM environment, the equation is inherently formulated on the finite domain Γ, and no such difficulties arise.

As a consequence of points 1 and 3, FIEM has the capacity to produce solutions of unparalleled accuracy; even on complicated problems, relative errors of 10^{-10} or less can be obtained.

It is worth emphasizing that FIEM is fast not only in the theoretical sense that it scales linearly with the complexity of the problem; the constant of proportionality is often small. For BVP's involving constant coefficient operators, and no body loads, FIEM can be orders of magnitude

faster than other methods. The only real competitor in this environment are methods based on the FFT (or spectral methods more generally), but such methods are limited in that they typically achieve high performance only on simple domains. They also tend to be limited in the extent to which they can handle adaptivity, and non-uniform grids more generally.

1.3. Generalizations. FIEM has a profound competitive advantage when applied to BVPs that can be reformulated as integral equations defined on the boundary. With some exceptions, this happens when the partial differential operator has constant coefficients, and when there is no body load. However, it is occasionally advantageous to use FIEM for problems involving body loads, and / or non-constant coefficients as well. In such environments, one must use integral equations that involve volume integrals which implies a higher computational cost. However, the unrivaled stability and accuracy of an integral equation formulation can sometimes justify the cost; for instance when dealing with scattering problems close to resonant frequencies. These matters are discussed further in Sections **??** and **??**.

1.4. When to use FIEM. Generally speaking, FIEM works for elliptic linear BVPs involving partial differential operators whose highest order term has constant coefficients. However, they are particularly appealing for situations involving any of the following:

- Constant coefficient differential operators.
- No body load.
- Exterior problems defined on infinite domains.
- Relatively ill-conditioned physics (thin domains, scattering at short wave-lengths, etc.)
- Very high accuracy is required.

As a more practical consideration, the set-up costof using an FIEM is typically higher than that of using say a finite element discretization so in situations where you are only interested in a one-off solve at moderate accuracy, FIEM might not be the best choice. (The hardest part is usually to find a good integral equation formulation. In most standard situations, this has already been done, but if you have unusual boundary conditions, multiple types of physics going on, strange boundaries in 3D, etc., then finding the right integral equation could be *a lot* of work.)

In addition to the "fundamental limitations" described in the previous paragraph, there is also an issue of availability of software. There is any number of software packages available for solving PDEs using finite element, finite difference, or finite volume methods — some commercial and some free. Some packages are very user friendly with GUIs, the ability to import geometry information from a CAD program, the ability to handle[2] almost any differential equation thrown at them, etc. In contrast, there are relatively few packages available for discretizing and solving an equation via FIEM, and what is available is often raw Fortran code.

1.5. Outline of chapter.

2. Notation, orientation of boundaries, etc

Unless otherwise noted, Ω will in this section denote a simply connected boundary domain in the plane with smooth boundary Γ. For $\boldsymbol{x} \in \Gamma$, $\boldsymbol{n}(\boldsymbol{x})$ denotes a unit normal that points out from Ω. We let Ψ denote the domain exterior to Γ. See Figure 0.1 for an illustration.

[2]The term "handle" does not necessarily imply that the output is particularly close to the correct solution

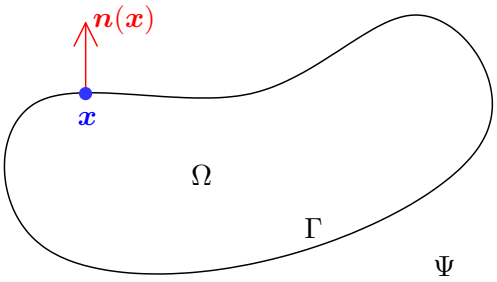

FIGURE 0.1. Notation for geometries. Γ is a smooth contour in the plane. Ω is the domain interior to Γ and Ψ is the exterior domain. $\boldsymbol{n}(\boldsymbol{x})$ denotes the outwards pointing unit normal.

3. A model problem

In this section, we demonstrate how FIEM can be used to solve a specific problem: The Laplace equation on a bounded smooth domain in $\mathbb{R}^2$ with Dirichlet boundary data. The objective is to demonstrate the entire solution procedure for a particularly simple example. We will then return to different points in later sections and describe how to generalize the techniques to other equations and to other geometries.

3.1. Reformulation of a Boundary Value Problem as a Boundary Integral Equation. We consider the BVP

$$-\Delta u = 0 \qquad \text{on } \Omega, \tag{5}$$

$$u = f \qquad \text{on } \Gamma, \tag{6}$$

where Ω is a simply connected open set in $\mathbb{R}^2$ with smooth boundary Γ, and where f is a prescribed function on Γ. We seek to rewrite (5, 6) as an equation defined on Γ only. To this end, we make the Ansatz

$$u(x) = \int_\Gamma \frac{n(x') \cdot (x - x')}{2\pi |x - x'|^2} \sigma(x')\, dl(x'), \qquad \text{for } x \in \Omega, \tag{7}$$

where $n(x')$ is the unit outwards pointing normal of Γ at x', and where σ is an unknown function on Γ. Note that for any σ, the function u defined by (7) satisfies (5).[3] Equation (6) is satisfied if and only if the function u defined by (7) satisfies

$$\lim_{y \to x,\, y \in \Omega} u(y) = f(x), \qquad \forall\, x \in \Gamma. \tag{8}$$

A simple evaluation of the limit (8) (see Section 5) yields

$$-\frac{1}{2}\sigma(x) + \int_\Gamma \frac{n(x') \cdot (x - x')}{2\pi |x - x'|^2} \sigma(x')\, dl(x') = f(x), \qquad \text{for } x \in \Gamma. \tag{9}$$

In a strong sense, the BVP (5, 6) is equivalent to the BIE (9).

[3] Conversely, it is known from classical potential theory that any solution of (5, 6) can be written in the form (7) for some function σ.

3.2. Parameterization of BIE. We next seek to approximate the BIE (9) by a set of linear algebraic equations. To this end, we parameterize Γ via a function $G(t) = (G_1(t),\, G_2(t))$ such that

$$\Gamma = \{G(t) :\, t \in [0,\, T]\}. \tag{10}$$

Causing hopefully not too much confusion, we think of any functions defined on Γ as functions of the parameter t:[4]

$$u = u(t), \qquad \sigma = \sigma(t), \qquad f = f(t).$$

We can also express the double layer kernel in terms of parameter values:

$$\begin{aligned} D(t,t') &= \frac{n(t')\cdot\big(G(t)-G(t')\big)}{2\pi|G(t)-G(t')|^2} \\ &= \frac{G_2'(t')\,\big(G_1(t)-G_1(t')\big) - G_1'(t')\,\big(G_2(t)-G_2(t')\big)}{2\pi\,\big((G_1'(t'))^2+(G_2'(t'))^2\big)^{1/2}\,\big((G_1(t)-G_1(t'))^2+(G_2(t)-G_2(t'))^2\big)}, \end{aligned} \tag{11}$$

where we used that

$$n(t') = \frac{\big(G_2'(t'),\, -G_1'(t')\big)}{(G_1'(t'))^2+(G_2'(t'))^2}.$$

Inserting the parameterization into (9) we obtain

$$-\frac{1}{2}\sigma(t) + \int_0^T D(t,t')\,\sigma(t')\,|G'(t')|\,dt' = f(t), \qquad \text{for } \tau \in [0,\, T]. \tag{12}$$

(Observe that the formulas can be simplified slightly since the term $|G'(t')|$ appears in the denominator of $D(t,t')$. This would be beneficial for now, but would cause problems later when we consider more general quadratures.)

REMARK 0.3. *In most cases, the kernel in a BIE is singular as $t' \to t$, and at first glance it appears the the kernel $D(t,t')$ in (11) is singular as well. However, it turns out that when the contour is smooth, we have $n(x')\cdot(x-x') = O(|x-x'|^2)$ since the angle between $n(x')$ and $x-x'$ approach $\pi/2$. In consequence, the singularities in the numerator and the denominator precisely cancel, and the kernel is in fact a smooth function. Using l'Hôpital's rule twice, we find that*

$$\lim_{t'\to t} D(t,t') = \frac{G_2'(t)\,G_1''(t) - G_1'(t)\,G_2''(t)}{4\pi|G'(t)|^3}.$$

3.3. Discretization of BIE. We discretize (12) using the *Nyström method*, as follows: First we discretize the integral in (12) using a quadrature rule with weights $(w_j)_{j=1}^N$, and quadrature points $(t_j)_{n=1}^N \subset [0,\, T]$, and obtain the approximate equation

$$-\frac{1}{2}\sigma(t) + \sum_{j=1}^N w_j\, D(t,t_j)\,|G'(t_j)|\,\sigma(t_j) = f(t), \qquad \text{for } t \in [0,\, T]. \tag{13}$$

To obtain a discrete equation from (13), we require that (13) should hold for at the quadrature points, *i.e.* when $t = t_1, \ldots, t_N$. This results in the system of equations

$$-\frac{1}{2}\sigma(t_i) + \sum_{j=1}^N w_j\, D(t_i,t_j)\,|G'(t_j)|\,\sigma(t_j) = f(t_i), \qquad \text{for } i = 1,\, 2,\, \ldots,\, N. \tag{14}$$

[4]To be mathematically rigorous, we should have defined a new function $U = u \circ G$ (so that $U(t) = u(G(t))$), a new function $\Sigma = \sigma \circ G$, and a new function $F = f \circ G$. Then u, σ, and f would have been functions defined on Γ, and U, Σ, and F would have been functions defined on $[0,\, T]$.

We write (14) in matrix form as

$$\left(-\frac{1}{2}\mathbf{I} + \mathbf{B}\right)\boldsymbol{\sigma} = \mathbf{f}, \tag{15}$$

where $\mathbf{f}$ is a vector of pointwise samples from f, $\mathbf{f}(i) = f(t_i)$, and where $\boldsymbol{\sigma}$ is a vector approximating the exact solution σ, $\boldsymbol{\sigma}(i) \approx \sigma(t_i)$, and where $\mathbf{B}$ has the entries

$$\mathbf{B}(i,j) = D(t_i, t_j)\,|G'(t_j)|\,w_j.$$

Finally, we note that since the integrand in this particular case is smooth and periodic, the trapezoidal rule ($w_i = T/N$, $t_i = i\,T/N$) achieves superpolynomial convergence rates; and find that

$$\mathbf{B}(i,j) = \frac{T}{N}\,D(t_i, t_j)\,|G'(t_j)|.$$

3.4. Post-processing: The numerical method described so far yields as its result the dipole distribution σ, which is generally not the quantity sought. However, any other quantity of interest can be computed via the formulas:

$$u(x) = -\frac{1}{2}\sigma(x) + \int_\Gamma D(x,x')\,\sigma(x')\,dl(x'), \qquad x \in \Gamma \tag{16}$$

$$u(x) = \int_\Gamma D(x,x')\,\sigma(x')\,dl(x'), \qquad x \in \Omega. \tag{17}$$

The kernel in the first formula is smooth and it can easily be evaluated upon discretization via the trapezoidal rule. The kernel in the second formula can also be evaluated very easily whenever the target point x is not that close to the boundary. However, if x is close to the boundary, then $D(x,x')$ has very sharp derivatives when x' approaches x and some care must be exercise.

The evaluations of the layer potentials (16) and (17) can be accelerated by the Fast Multipole Method if the potential is sought at a large number of target points.

4. Review of the Green identities

Recall that if $\boldsymbol{f}$ is a vector valued C^1 function on Ω, then the Gauss theorem reads

$$\int_\Omega \nabla\cdot\boldsymbol{f} = \int_\Gamma \boldsymbol{n}\cdot\boldsymbol{f},$$

where $\boldsymbol{n}$ is the outward pointing unit normal to Γ. Now suppose that u and v are two C^2 scalar value functions. Applying the Gauss theorem to $\boldsymbol{f} = u\nabla v$ and $\boldsymbol{f} = (\nabla u)v$ we find

$$\int_\Omega \big(u\,\Delta v + \nabla u\cdot\nabla v\big) = \int_\Gamma u\,\frac{\partial v}{\partial n} \tag{18}$$

$$\int_\Omega \big(\Delta u\,v + \nabla u\cdot\nabla v\big) = \int_\Gamma \frac{\partial u}{\partial n}\,v. \tag{19}$$

Subtracting (19) from (18), we find

$$\int_\Omega \big(u\,\Delta v - \Delta v\,u\big) = \int_\Gamma \left(u\,\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n}\,v\right).$$

Now fix a point $x \in \Omega$ and set

$$v(y) = -\frac{1}{2\pi}\log|y-x|.$$

Further, assume u satisfies

$$-\Delta u = 0 \qquad \text{in } \Omega,$$

let ε be a small number (smaller than $\operatorname{dist}(\Gamma, x)$), set

$$\Omega_\varepsilon = \Omega \backslash B_\varepsilon(x)$$

and observe that

$$\partial\Omega_\varepsilon = \Gamma \cup \Gamma_\varepsilon$$

where

$$\Gamma_\varepsilon = \partial B_\varepsilon(x).$$

Now note that $\Delta u(y) = \Delta v(y)$ for $y \in \Omega_\varepsilon$ to obtain

$$0 = \int_\Gamma \left(u \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} v \right) + \int_{\Gamma_\varepsilon} \left(u \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} v \right). \tag{20}$$

We next seek to evaluate the limit as $\varepsilon \to 0$. For $y \in \Gamma_\varepsilon$, we find

$$\begin{aligned} v(y) =& -\frac{1}{2\pi} \log|x-y| = -\frac{1}{2\pi} \log \varepsilon \\ \frac{\partial v}{\partial n}(y) =& n(y) \cdot \nabla v(y) = \frac{x-y}{|x-y|} \frac{x-y}{2\pi|x-y|^2} = \frac{1}{2\pi|x-y|} = \frac{1}{2\pi\varepsilon}. \end{aligned}$$

Since $u \in C(\Omega)$ we find

$$\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} u(y) \frac{\partial v}{\partial n}(y)\, dl(y) = \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} u(y) \frac{1}{2\pi\varepsilon}\, dl(y) = u(x).$$

Since $u \in C^1(\Omega)$ we know that for some finite M, we have $|\nabla u(y)| \le M$ and so

$$\lim_{\varepsilon\to 0} \left| \int_{\Gamma_\varepsilon} \frac{\partial u}{\partial n}(y)\, v(y)\, dl(y) \right| \le \limsup_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} M \frac{1}{2\pi} |\log\varepsilon|\, dl(y) = \limsup_{\varepsilon\to 0} M\, \varepsilon\, |\log\varepsilon| = 0.$$

Taking the limit $\varepsilon \to 0$ in (20) we obtain

$$u(x) = \int_\Gamma \frac{n(y)\cdot(y-x)}{2\pi|y-x|^2} u(y)\, dl(y) - \int_\Gamma \frac{1}{2\pi} \log|y-x|\, u_n(y)\, dl(y). \tag{21}$$

We introduce two integral operators called *layer potentials* via

$$\textit{The single layer operator:} \qquad [S\,w](x) = \int_\Gamma \frac{-1}{2\pi} \log|x-y|\, w(y)\, dl(y), \tag{22}$$

$$\textit{The double layer operator:} \qquad [D\,w](x) = \int_\Gamma \frac{n(y)\cdot(x-y)}{2\pi|x-y|^2} w(y)\, dl(y). \tag{23}$$

Then (21) takes the form

$$u(x) = -[D\,u](x) + [S\,u_n](x) \qquad x \in \Omega. \tag{24}$$

An entirely analogous computation shows that

$$\frac{1}{2} u(x) = -[D\,u](x) + [S\,u_n](x) \qquad x \in \Gamma. \tag{25}$$

5. Derivation of BIEs for Laplace's equation

5.1. The direct formulation. We first derive a BIE for the Dirichlet problem

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega, \\ u(x) = g(x), & x \in \Gamma. \end{cases}$$

From (25) we find that

$$\frac{1}{2}\, g(x) + [D\, g](x) = [S\, u_n](x) \qquad x \in \Gamma. \tag{26}$$

Equation (27) can be solved for the unknown boundary function u_n. Once u_n has been determined, the potential u can be recovered from (24).

Next consider the Neumann problem

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega, \\ u_n(x) = h(x), & x \in \Gamma. \end{cases}$$

From (25) we find that

$$\frac{1}{2}\, u(x) + [D\, u](x) = [S\, h](x) \qquad x \in \Gamma. \tag{27}$$

Equation (27) can be solved for the unknown boundary function $u|_\Gamma$. Once $u|_\Gamma$ has been determined, the potential u can be recovered from (24).

REMARK 0.4. *The so called "direct" formulation for works very well for the Neumann problem since the operator $(1/2)I + D$ is an invertible second kind Fredholm operator. It works less well for the Dirichlet problem since S is compact, and the inverse of a compact operator is unbounded. The difficulties involved can all be resolved, but it takes some effort.*

5.2. The indirect formulation. Consider the Dirichlet problem

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega, \\ u(x) = g(x), & x \in \Gamma. \end{cases}$$

We look for a solution of the form

$$u(x) = [D\sigma](x),$$

where σ is an unknown boundary potential. A well-conditioned equation for σ is obtained from the formula

$$\lim_{z \in \Omega,\ z \to x} [D\sigma](z) = -\frac{1}{2}\sigma(x) + [D\sigma](x), \qquad x \in \Gamma.$$

We find

$$-\frac{1}{2}\sigma(x) + [D\sigma](x) = g(x).$$

Next consider the Neumann problem

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega, \\ u_n(x) = h(x), & x \in \Gamma. \end{cases}$$

We look for a solution of the form

$$u(x) = [S\sigma](x),$$

where σ is an unknown boundary potential. For $x \in \Gamma$, we find

$$u_n(x) = \lim_{z\in\Omega,\ z\to x} n(x)\cdot\nabla[S\sigma](z) = \frac{1}{2}\sigma(x) + [D^*\sigma](x),$$

where D^* is the *adjoint* of the double layer potential,

$$[D^*\sigma](x) = \int_\Gamma \frac{-n(x)\cdot(x-y)}{2\pi\,|x-y|^2}\,ds(y).$$

(Observe that the normal is evaluated at the target point x, not at the integration variable y.) We find that the equation for σ reads

$$\frac{1}{2}\sigma(x) + [D^*\sigma](x) = h(x), \qquad x\in\Gamma. \tag{28}$$

The operator $(1/2)I + D^*$ is a second kind Fredholm operator, but it has a one-dimensional null-space, and a one-dimensional co-range. This is very natural, since we know (1) that the Neumann problem does not determine u other than up to translation, and (2) that the Neumann problem is not well-posed unless

$$\int_\Gamma h(x)\,ds(x) = 0.$$

Some care must be exercised in solving (28). An easy way of handling this is to modify the Ansatz to read

$$u(x) = [S\sigma](x) - \frac{1}{2\pi}\log|x-\hat{x}|\int_\Gamma \sigma(x)\,ds(x), \tag{29}$$

where $\hat{x}$ is a point chosen in the interior of Ω. Then the new equation for σ is

$$\frac{1}{2}\sigma(x) + [D^*\sigma](x) - \frac{n(x)\cdot(x-\hat{x})}{2\pi|x-\hat{x}|^2}\int_\Gamma \sigma(x)\,ds(x) = h(x), \qquad x\in\Gamma. \tag{30}$$

Integrating (30) over Γ, we find

$$\underbrace{\int_\Gamma \left(\frac{1}{2}\sigma(x) + [D^*\sigma](x)\right) ds(x)}_{=0} - \underbrace{\int_\Gamma \frac{n(x)\cdot(x-\hat{x})}{2\pi|x-\hat{x}|^2}\,ds(x)}_{=-1}\int_\Gamma \sigma(x)\,ds(x) = \underbrace{\int_\Gamma h(x)\,ds(x)}_{=0}.$$

In other words, when (30) holds, we have

$$\int_\Gamma \sigma(x)\,ds(x) = 0,$$

which implies that the "new" term in (29) vanishes.

REMARK 0.5. *The exterior Neumann problem is easier to handle. Consider the equation*

$$\begin{cases} -\Delta u(x) = 0, & x\in\Psi, \\ \quad u_n(x) = h(x), & x\in\Gamma. \end{cases} \tag{31}$$

Note that we do not need h to integrate to zero for (31) to have a solution. We still have a problem in that the solution is not uniquely determined unless we specify some decay condition at infinity. Observing that the total charge that generates u is $\int_\Gamma h$, it is natural to require u to satisfy

$$u(x) = \frac{1}{2\pi}\log\frac{1}{|x|}\int_\Gamma h(x')\,ds(x') + O(1/|x|), \qquad \text{as } |x|\to\infty.$$

Formally, we couple (31) with

$$\lim_{R\to\infty}\sup_{|x|=R}\left|u(x) - \frac{1}{2\pi}\log\frac{1}{|x|}\int_\Gamma h(x')\,ds(x')\right| = 0. \tag{32}$$

Condition (32) is automatically satisfied if we look for a solution of the form

$$u(x) = [S\sigma](x),$$

where σ is an unknown boundary potential. The relevant equation for σ is now

$$-\frac{1}{2}\sigma(x) + [D^*\sigma](x) = h(x), \qquad x \in \Gamma. \tag{33}$$

The operator $-(1/2)I + D^*$ is a second kind Fredholm operator that is onto and one-to-one.