WRONSKIANS

In general, it can be hard to determine whether a given set of functions $\{f_1, f_2, \ldots, f_n\}$ is linearly dependent.

One technique that sometimes help is to form the "Wronskian".

If the functions all have $n-1$ derivatives, then define the "Wronskian" by

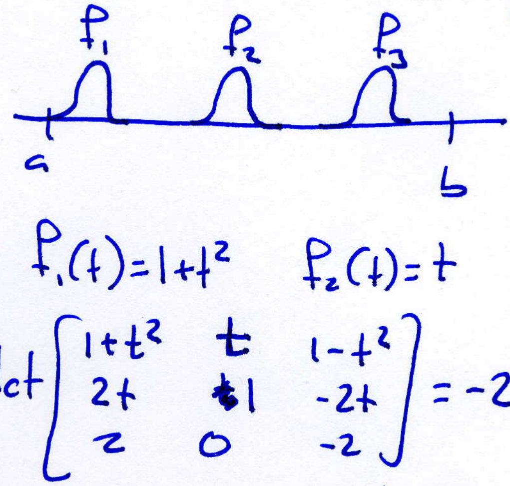
$$W(t) = \det \begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & & f_n'(t) \\ f_1''(t) & f_2''(t) & & f_n''(t) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{bmatrix}$$

Thm If a set $\{f_1, f_2, \ldots, f_n\}$ of functions in $C^{(n-1)}(I)$ is linearly dependent, then $W(t) = 0$ for all $t \in I$.

Corollary If for some $t \in I$ we have $W(t) \neq 0$, then $\{f_1, f_2, \ldots, f_n\}$ is a linearly independent set.

Important: If ~~d~~ $W(t) = 0$ for all t, the test is inconclusive!
It is not necessarily the case that the set is linearly dep.

Example



$\text{support}(f_i) \cap \text{support}(f_j) = \emptyset$
$\Rightarrow W(t) = 0$ but $\{f_1, f_2, f_3\}$ is an independent set!

Example $f_1(t) = 1 + t^2 \quad f_2(t) = t \quad f_3(t) = 1 - t^2 \quad S = \{f_1, f_2, f_3\}$.

$$W(t) = \det \begin{bmatrix} 1+t^2 & t & 1-t^2 \\ 2t & 1 & -2t \\ 2 & 0 & -2 \end{bmatrix} = -2(1+t^2) - 4t^2 + 4t^2 - 2(1-t^2) = -4$$

Since $W(t) \neq 0$, S must be a linearly independent set.