Below you will find some extra exercises for practising separation of variables, and changes of variables. (These problems are **not** part of the regular home work and will not be collected.)

You will learn *far* more from these problems if you try to solve them without looking at the solutions first!!!

Example 1: Solve the ODE

$$y' = y^2 - 4.$$

Example 2: Solve the ODE

$$y' = y + e^t\, y^2.$$

Hint: Try the substitution $y(t) = e^t\, u(t)$.

Example 3: Solve the ODE

$$y' = t^2\, e^y - \frac{2}{t}.$$

Hint: Try the substitution $u(t) = t^2\, e^{y(t)}$.

Example 1: Solve the ODE

$$y' = y^2 - 4.$$

Solution: First we note that $y = 2$ and $y = -2$ are the constant solutions. In the remainder of the solution, we assume that $y \neq \pm 2$. Then $y^2 - 4 \neq 0$ and we get the equation

$$\frac{dy}{y^2 - 4} = dt \tag{1}$$

Note that

$$\frac{1}{y^2-4} = \frac{1}{(y-2)(y+2)} = \frac{1/4}{y-2} - \frac{1/4}{y+2}.$$

Integrating (1) we then obtain

$$\frac{1}{4}\log|y-2| - \frac{1}{4}\log|y+2| = t + C$$

$$\log\left|\frac{y-2}{y+2}\right| = 4\,t + 4\,C$$

$$\left|\frac{y-2}{y+2}\right| = e^{4\,C}\,e^{4\,t}$$

Set $D = \pm e^{4\,C}$. Then

$$\frac{y-2}{y+2} = D\,e^{4\,t}$$

$$y\,(1 - D\,e^{4\,t}) = 2\,(1 + D\,e^{4\,t})$$

$$y = 2\,\frac{1 + D\,e^{4\,t}}{1 - D\,e^{4\,t}}$$

Answer: The solutions are $y = 2$, $y = -2$, and for any non-zero real number D, the function

$$y = 2\,\frac{1 + D\,e^{4\,t}}{1 - D\,e^{4\,t}}. \tag{2}$$

Extra problems to be solved using a computer:

- Draw the direction field of the equation.
- Mark the two constant solutions.
- Fix a negative number D (say $D = -2$) and draw the function (2). What can you say about $\lim_{t\to\infty} y(t)$?
- Fix a different negative value of D and draw this function as well.
- Fix a couple of different positive numbers D and draw the functions (2). Something interesting happens around $t = -(1/4)\log(D)$ — what?
- What can you say about $\lim_{t\to\infty} y(t)$ when D is a positive number?

Example 2: Solve the ODE

$$y' = y + e^t\, y^2.$$

Hint: Try the substitution $y(t) = e^t\, u(t)$.

Solution: We have

$$y' = e^t\, u + e^t\, u'.$$

Then the equation takes the form

$$e^t\, u + e^t\, u' = e^t\, u + e^{3\,t}\, u^2,$$

which simplifies to

$$u' = e^{2\,t}\, u^2.$$

The right hand side is zero when $u = 0$. We note that $u = 0$ corresponds to the constant solution $y = 0$. Now suppose that $u \neq 0$. Then

$$\int \frac{du}{u^2} = \int e^{2\,t}\, dt$$

and so

$$-\frac{1}{u} = \frac{1}{2}\, e^{2\,t} + C.$$

We get

$$u = -\frac{1}{C + \frac{1}{2}\, e^{2\,t}}.$$

Finally we convert back to the function y:

$$y = e^t\, u = -\frac{e^t}{C + \frac{1}{2}\, e^{2\,t}}.$$

Answer: The solutions are $y = 0$, and for any real number C, the function

(3) $$y = -\frac{e^t}{C + \frac{1}{2}\, e^{2\,t}}.$$

Example 3: Solve the ODE

$$y' = t^2\, e^y - \frac{2}{t}.$$

Hint: Try the substitution $u(t) = t^2\, e^{y(t)}$.

Solution: We have

$$u' = 2\, t\, e^y + t^2\, y'\, e^y.$$

Note that the equation is not defined for $t \neq 0$! We can therefore write

$$u' = \frac{2u}{t} + u\, y'.$$

We also have

$$y' = t^2\, e^y - \frac{2}{t} = u - \frac{2}{t}.$$

Combining the two equations above we obtain an equation for u:

$$u' = \frac{2\, u}{t} + u\left(u - \frac{2}{t}\right) = u^2.$$

Now that we have a separable equation for u we proceed as usual. Note that since $t \neq 0$, it must be the case that $u \neq 0$, so we can divide by u^2 and obtain the equation

$$\int \frac{du}{u^2} = \int dt$$

$$-\frac{1}{u} = t + C$$

Set $D = -C$, so that

$$u = \frac{1}{D - t}$$

Finally we convert back to y:

$$y = \log\left(\frac{u}{t^2}\right) = \log\left(\frac{1}{t^2(D-t)}\right).$$

Answer: Any solution of the equation takes the form

$$y = \log\left(\frac{1}{t^2(D-t)}\right). \tag{4}$$

where D is any real number.

Extra problem: Differentiate the given solution to verify that the calculation is correct!

Extra problem: Plot the solution for a few different values of D. For any given D, determine for which values of t the solution is defined.