APPM 2360: Final Exam — Solutions
7.30 pm – 10.00 pm, May 3, 2008.

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section, (4) your instructor's name, and (5) a grading table. Show ALL of your work in your bluebook and box in your final answer. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

Problem 1: *(14 points, 2 each)* State whether the following statements are *always* "TRUE" or "FALSE" (meaning not *always* true). You MUST write the full word TRUE or FALSE. T/F or YES/NO will NOT be given any credit. You do not have to justify your answer for this question.

(a) If A and B are invertible matrices and X is a matrix such that $AX = B$, then X is also invertible.

(b) The transpose of a matrix exists if and only if the matrix is square.

(c) If A is a 3×3 matrix that is row equivalent to the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then A has rank 3.

(d) The system $\begin{cases} \dot{x} = & -y, \\ \dot{y} = & x \end{cases}$ has an asymptotically stable equilibrium point.

(e) If y_1 and y_2 are two different solutions of the equation $y' + y^2 = 1$, then the function $z = y_1 - y_2$ satisfies $z' + z^2 = 0$.

(f) If f, g, and h are continuously differentiable functions on $\mathbb{R}$ such that $f' = g$, then the function $y(t) = \int_0^t e^{-f(t)+f(s)}\, h(s)\, ds$ satisfies $y' + g\,y = h$.

(g) If the matrix A has eigenvalues 1 and 3 then the matrix $A - I$ has eigenvalues 0 and 2.

Solution:

(a) TRUE

(b) FALSE

(c) TRUE

(d) FALSE (the origin is stable but not asymptotically stable)

(e) FALSE (the equation is non-linear, consider for instance $y_1 = 1$, $y_2 = -1$, $z = 2$)

(f) TRUE (this is the formula for a particular solution from an integrating factor)

(g) TRUE

Problem 2: *(12 points, 6 each)* Consider the differential equation

$$y'' + 2\,y' + 5\,y = f(t)$$

(a) Construct the general solution of the equation when $f(t) = 0$.

(b) Construct the solution of the equation when $f(t) = e^{-t}$ subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$.

Solution:

(a) The characteristic equation is $r^2 + 2\,r + 5 = 0$. The roots are $r_{1,1} = -1 \pm 2\,i$. Consequently, the general solution of the homogeneous equation is

$$y(t) = e^{-t}\left(c_1\,\cos(2\,t) + c_2\,\sin(2\,t)\right),$$

where c_1 and c_2 are arbitrary real numbers.

(b) We use the technique of undetermined coefficients to find a particular solution. Since -1 is not a root of the characteristic equation, we make the Ansatz

$$y_{\mathrm{p}} = A\,e^{-1}.$$

Inserting this y_{p} into the equation we obtain

$$y_{\mathrm{p}}'' + 2\,y_{\mathrm{p}}' + 5\,y_{\mathrm{p}} = A\,(-1)^2\,e^{-t} + 2\,A\,(-1)\,e^{-t} + 5\,A\,e^{-t} = 4\,A\,e^{-t}.$$

The right hand side equals f if $A = 1/4$.

The general solution given in (a) is the solution of the homogeneous equation. The general solution of the inhomogeneous equation is then

$$y(t) = e^{-t}\left(c_1\,\cos(2\,t) + c_2\,\sin(2\,t)\right) + \frac{1}{4}\,e^{-t}.$$

Then

$$y(0) = c_1 + \frac{1}{4}$$

so the condition $y(0) = 0$ is satisfied by setting $c_1 = -\frac{1}{4}$. Further,

$$y'(t) = -e^{-t}\left(c_1\,\cos(2\,t) + c_2\,\sin(2\,t)\right) + e^{-t}\left(-2\,c_1\,\sin(2\,t) + 2\,c_2\,\cos(2\,t)\right) - \frac{1}{4}\,e^{-t}.$$

We find that

$$y'(0) = -c_1 + 2\,c_2 - \frac{1}{4} = \{\text{Recall: } c_1 = -\frac{1}{4}\} = 2\,c_2.$$

The condition $y'(0) = 1$ is then satisfied by setting $c_2 = \frac{1}{2}$.

Answer: $y(t) = e^{-t}\left(\frac{1}{4}\,(1 - \cos(2\,t)) + \frac{1}{2}\,\sin(2\,t)\right)$.

Problem 3: *(12 points, 2 each)* Match each linear system with the corresponding phase-plane. You do not have to justify your answer to this question.

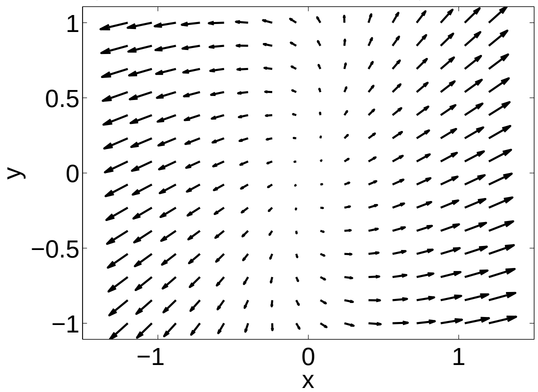
(i) $\begin{cases} x' = 3x - 2y \\ y' = 4x - y \end{cases}$ **(ii)** $\begin{cases} x' = -x + y \\ y' = -2x + y \end{cases}$

(iii) $\begin{cases} x' = 4x - y \\ y' = 2x + y \end{cases}$ **(iv)** $\begin{cases} x' = x - 2y \\ y' = -2x + 4y \end{cases}$

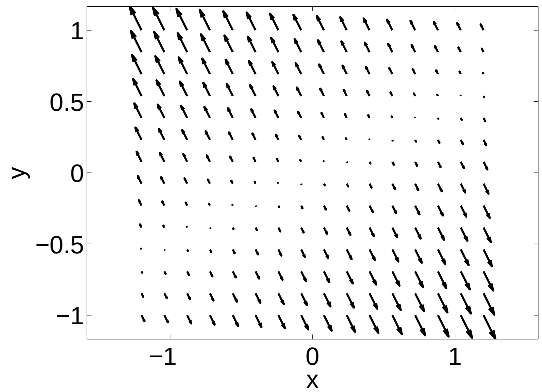
(v) $\begin{cases} x' = 2y \\ y' = -\sin(\pi x) \end{cases}$ **(vi)** $\begin{cases} x' = -2y \\ y' = -\sin(\pi x) \end{cases}$

Solution

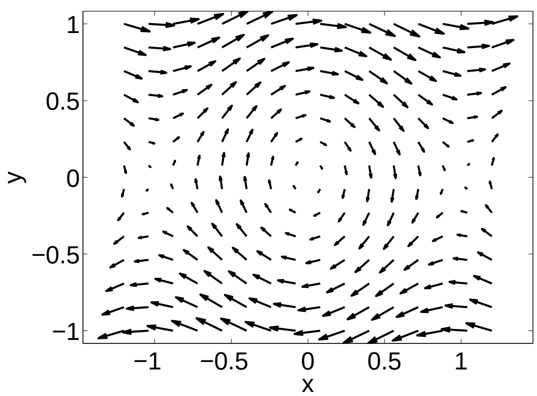
(a)	iii	(i)	e
(b)	iv	(ii)	f
(c)	v	(iii)	a
(d)	vi	(iv)	b
(e)	i	(v)	c
(f)	ii	(vi)	d



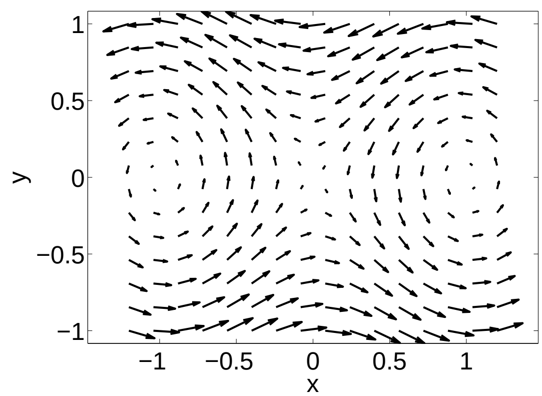
(a)



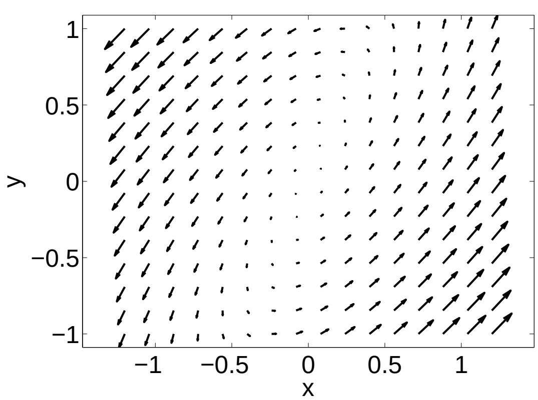
(b)



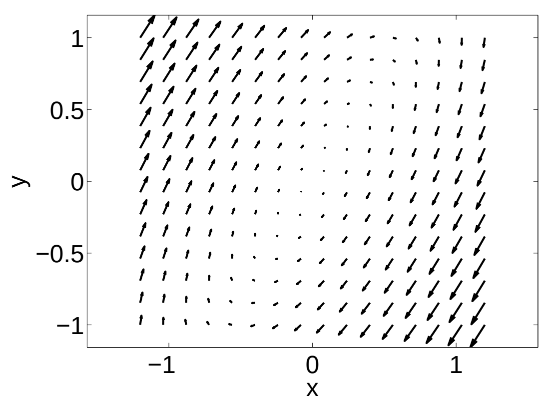
(c)



(d)



(e)



(f)

Problem 4: *(12 points, 4 each)* Let t_0 and y_0 be real numbers such that $t_0 > 0$ and $1 < y_0 < 2$, and consider for positive t the initial value problem (IVP)

$$y'(t) = \frac{1}{t}(y-1)(y-2)$$
$$y(t_0) = y_0.$$

(a) Show that for all positive values of t, $1 < y(t) < 2$.

(b) Show that if $t_1 < t_2$, then the solution of the IVP satisfies $y(t_1) > y(t_2)$.
Hint: $y(t_2) - y(t_1) = \int_{t_1}^{t_2} y'(t)\, dt$.

(c) Show that the solution of the IVP satisfies $\lim_{t \to 0} y(t) = 2$.

Solution: Set $f(t, y) = \frac{1}{t}(y-1)(y-2)$.

(a) First note that since both f and $\frac{\partial f}{\partial y}$ are continuous, (IVP) has a unique solution for every initial condition with $t_0 > 0$ (in other words, no trajectories may cross).

The lines $y = 1$ and $y = 2$ are equilibrium solutions of the ODE. Since no trajectories may cross each other, any solution that starts between them must stay between them for all positive t.

(b) We proved in (a) that $1 < y(t) < 2$ for all $t > 0$. This means that for any positive t, we have

$$y'(t) = f(t, y(t)) = \underbrace{\frac{1}{t}}_{>0} \underbrace{(y-1)}_{>0} \underbrace{(y-2)}_{<0} < 0.$$

It then follows that

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} \underbrace{y'(t)}_{<0}\, dt < 0.$$

(c) Since the differential equation is separable, we can solve it analytically:

$$\frac{dy}{dt} = \frac{1}{t}(y-1)(y-2) \quad \Rightarrow \quad \frac{dy}{(y-1)(y-2)} = \frac{dt}{t} \quad \Rightarrow \quad \int \left(\frac{1}{y-2} - \frac{1}{y-1}\right) dy = \int \frac{1}{t}\, dt$$
$$\Rightarrow \quad \log|y-2| - \log|y-1| = \log t + C \quad \Rightarrow \quad \left|\frac{y-2}{y-1}\right| = e^C\, t,$$

where C is a constant of integration. Setting $B = \pm e^C$ we get

$$\frac{y-2}{y-1} = B\, t,$$

and solving for y yields

$$y(t) = \frac{2 - B\, t}{1 - B\, t}.$$

It follows immediately that

$$\lim_{t \searrow 0} y(t) = \frac{2 - B\, 0}{1 - B\, 0} = 2.$$

Remark: Note that $B = \frac{1}{t_0} \frac{y_0 - 2}{y_0 - 1}$ so when $1 < y_0 < 2$, B is negative.

Problem 5: *(12 points, 4 each)* Consider the differential equation

$$w'(t) = w(t)^2 + P(t)\, w(t) + Q(t)$$

(a) Use the substitution $w = -\dfrac{y'}{y}$ to show that $y = y(t)$ satisfies $y'' - P(t)y' + Q(t)y = 0$.

(b) Take $P(t) = 2$, $Q(t) = 1$ and find the general solution of the equation for $w = w(t)$.

(c) For the same values of P and Q find the general solution of the equation for $y = y(t)$ and verify that the two solutions satisfy $y' + w\, y = 0$.

Solution:

(a) If $w = -y'/y$, then

$$w' = -\frac{y''}{y} + \frac{(y')^2}{y^2}.$$

It follows that

$$-\frac{y''}{y} + \frac{(y')^2}{y^2} = \left(-\frac{y'}{y}\right)^2 - P\,\frac{y'}{y} + Q.$$

Noting that the terms $(y'/y)^2$ cancel, we multiply both sides by y and obtain

$$-y'' = -P\, y' + Q\, y,$$

which is what we were asked to show.

(b) For $P = 2$, $Q = 1$ we find that y satisfies

$$y'' - 2\, y' + y = 0.$$

The characteristic equation is $r^2 - 2\, r + 1 = 0$. It has the double root $r = 1$, so

$$y(t) = c_1\, e^t + c_2\, t\, e^t.$$

Converting back to the original variable, we obtain

$$w(t) = -\frac{y'(t)}{y(t)} = -\frac{c_1\, e^t + c_2\, e^t + c_2\, t\, e^t}{c_1\, e^t + c_2\, t\, e^t} = -\frac{c_1 + c_2 + c_2\, t}{c_1 + c_2\, t}.$$

(c) We find that

$$w\, y = -\frac{c_1 + c_2 + c_2\, t}{c_1 + c_2\, t}\left(c_1\, e^t + c_2\, t\, e^t\right) = -\left(c_1 + c_2 + c_2\, t\right) e^t.$$

Since

$$y' = c_1\, e^t + c_2\, e^t + c_2\, t\, e^t,$$

we see that $w\, y = -y'$.

Remark: An alternative solution to part (b) is to directly solve the equation

$$w' = w^2 + 2\, w + 1$$

by exploiting that it is separable:

$$\frac{dw}{(w+1)^2} = dt \quad \Rightarrow \quad -\frac{1}{w+1} = t + C \quad \Rightarrow \quad w = -1 - \frac{1}{t + C}.$$

Setting $C = c_1/c_2$ one then easily verifies that $w\, y = -y'$.

Problem 6: *(12 points, 4 each)* Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -5 \end{bmatrix},$$

and the matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$.

(a) Find all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $A\,\mathbf{x} = 0$.

(b) Show that the set $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$ is not a basis for the linear subspace $\mathbb{V} = \operatorname{span}\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$ and provide a basis for $\mathbb{V}$.

(c) Determine whether the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ is in $\mathbb{V}$.

Solution:

(a) First we calculate the RREF of A:

$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \\ -3 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 3 \\ -3 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that x_3 is the free variable. Setting $x_3 = t$, it follows that $x_1 = -2\,t$ and $x_2 = t$. The general solution is then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2\,t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

(b) We determined in part (a) that $A \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = 0.$

An equivalent statement is that $-2\,\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 0$.
It follows that the set $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$ is not linearly independent.

From the calculation in part (a), we also found that the first and the second columns are pivot columns. This means that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, and consequently that $\boxed{\{\mathbf{v}_1,\, \mathbf{v}_2\} \text{ is a basis for } \mathbb{V}.}$

(c) $\mathbf{v} \in \mathbb{V}$ iff the equation $A\,\mathbf{x} = \mathbf{b}$ has a solution. Forming the REF of $[A|\mathbf{b}]$ we find that

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 2 \\ 2 & 1 & 3 & 0 \\ 1 & 1 & 1 & 1 \\ -3 & -1 & -5 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & 3 & 0 \\ -3 & -1 & -5 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -2 \\ 0 & 2 & -2 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The last line shows that the system cannot have a solution, and consequently $\boxed{\mathbf{b} \notin \mathbb{V}.}$

Remark: In part (c), it is more efficient to use the basis determined in part (b). For instance, using the basis $\{\mathbf{v}_1,\, \mathbf{v}_2\}$, one would show that the system $[\mathbf{v}_1\ \mathbf{v}_2 \mid \mathbf{b}]$ is inconsistent.

Problem 7: *(12 points, 6 each)* Consider the matrices A and B given by

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ -1 & -1 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -6 & 5 & t \\ s & -2 & 3 \\ -1 & 1 & -1 \end{bmatrix}.$$

(a) Find the real numbers s and t such that $A\,B = I$.

(b) Solve the following equation for $\mathbf{x}$:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ -1 & -1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} \pi \\ 9 \\ \sqrt{5} \end{bmatrix}.$$

Solution:

(a) Set $C = A\,B$. Then, for instance,

$$c_{21} = [0\ 1\ 3] \begin{bmatrix} -6 \\ s \\ -1 \end{bmatrix} = s - 3,$$

and

$$c_{13} = [1\ 2\ \ -1] \begin{bmatrix} t \\ 3 \\ -1 \end{bmatrix} = t + 6 + 1 = t + 7.$$

For C to be the identity matrix, c_{21} and c_{13} must both be zero. It follows that $\boxed{s = 3 \text{ and } t = -7.}$

(b) Since the coefficient matrix in the system is A, the solution is

$$\mathbf{x} = A^{-1} \begin{bmatrix} \pi \\ 9 \\ \sqrt{5} \end{bmatrix} = B \begin{bmatrix} \pi \\ 9 \\ \sqrt{5} \end{bmatrix} = \begin{bmatrix} -6 & 5 & -7 \\ 3 & -2 & 3 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \pi \\ 9 \\ \sqrt{5} \end{bmatrix} = \begin{bmatrix} -6\,\pi + 45 - 7\sqrt{5} \\ 3\,\pi - 18 + 3\sqrt{5} \\ -\pi + 9 - \sqrt{5} \end{bmatrix}.$$

Problem 8: *(14 points)* Let h be a real number and consider the nonlinear system

$$\begin{cases} \dot{x} = h\,x + y - x\,(x^2 + y^2) \\ \dot{y} = -x + h\,y - y\,(x^2 + y^2). \end{cases}$$

(a) *(6 points)* Linearize this system about the origin and discuss the stability of the system at that point for different values of h.

(b) *(4 points)* Make the substitution $x = r\cos\theta$ and $y = r\sin\theta$ when $h \neq 0$ and show that the new functions $r = r(t)$ and $\theta = \theta(t)$ satisfy

$$\begin{cases} \dot{r} = r\,(h - r^2) \\ \dot{\theta} = -1 \end{cases}$$

You may use the following formulas:

$$r\dot{r} = x\dot{x} + y\dot{y} \quad \text{and} \quad \dot{\theta} = \frac{x\dot{y} - \dot{x}y}{x^2 + y^2}.$$

(c) *(4 points)* Prove that if $h \leq 0$, then any solution $r(t)$ satisfies $r(t) \to 0$ as $t \to \infty$, whereas if $h > 0$, then any solution except the zero solution itself satisfies $r(t) \to \sqrt{h}$ as $t \to \infty$.

Solution: Set $f(x, y) = h\,x + y - x\,(x^2 + y^2)$ and $g(x, y) = -x + h\,y - y\,(x^2 + y^2)$.

(a) The Jacobian matrix is $J = \begin{bmatrix} h & 1 \\ -1 & h \end{bmatrix}$ with characteristic equation $(h - \lambda)^2 + 1 = 0$. The eigenvalues are then $\lambda_{1,2} = h \pm i$.

Since $\mathrm{Re}(\lambda_{1,2}) = h$, we find that the origin is **stable** when $h < 0$ and unstable when $h > 0$. When $h = 0$ we find that $\lambda_{1,2} = \pm i$ which is the case in which linear analysis is inconclusive (so in this case we cannot say anything).

(b) We have

$$\begin{aligned} r\,\dot{r} = x\,\dot{x} + y\,\dot{y} &= h\,x^2 + x\,y - x^2\,(x^2 + y^2) - x\,y + h\,y^2 - y^2\,(x^2 + y^2) \\ &= h\,(x^2 + y^2) - (x^2 + y^2)\,(x^2 + y^2) = h\,r^2 - r^4. \end{aligned}$$

Dividing by r we find that $\dot{r} = h\,r - r^3 = r\,(h - r^2)$. Further,

$$\dot{\theta} = \frac{x\dot{y} - \dot{x}y}{x^2 + y^2} = \frac{-x^2 + h\,x\,y - x\,y\,(x^2 + y^2) - h\,x\,y - y^2 + x\,y\,(x^2 + y^2)}{x^2 + y^2} = \frac{-x^2 - y^2}{x^2 + y^2} = -1.$$

(c) Suppose that $h \leq 0$. Then the equation $\dot{r} = r\,(h - r^2)$ has $r = 0$ as its only equilibrium point. Since $\dot{r} < 0$ when $r > 0$, this equilibrium point is stable, and so $r(t) \to 0$ as $t \to \infty$.

Suppose next that $h > 0$. Then the equation $\dot{r} = r\,(h - r^2)$ has two equilibrium points, $r = 0$ and $r = \sqrt{h}$. Since $\dot{r} > 0$ when $r \in (0, \sqrt{h})$ and $\dot{r} < 0$ when $r \in (\sqrt{h}, \infty)$, we find that $r = 0$ is an unstable equilibrium point, and $r = \sqrt{h}$ is a stable one. This proves that all solutions except the zero solution converge to $\pm\sqrt{h}$ as $t \to \infty$.

Remark: In the solution above, we follow standard practice for working with polar coordinates and assume that $r \geq 0$. However, the solution remains essentially unchanged if negative values of r are allowed. The only difference is that then $r = \pm\sqrt{h}$ are equilibrium points for the case $h > 0$. Both of these points are stable.