§4.3 / 4, 15, 19, 32, 43, 50, 63

4) Characteristic equation:

$$r^2 + 2r + 8 = 0$$

$$\Rightarrow \quad r = \frac{-2 \pm \sqrt{-28}}{2} = -1 \pm i\sqrt{7}$$

So the general solution is

$$y(t) = e^{-t}(c_1 \cos\sqrt{7}t + c_2 \sin\sqrt{7}t)$$

the basis for the solution space is

$$\mathcal{B} = \{ e^{-t}\cos\sqrt{7}t,\ e^{-t}\sin\sqrt{7}t \}$$

15) Characteristic equation:

$$r^2 - 4r + 7 = 0$$

$$\Rightarrow \quad r = \frac{4 \pm \sqrt{-12}}{2} = 2 \pm i\sqrt{3}$$

Then the general solution is

$$y(t) = e^{2t}(c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t)$$

From the initial conditions,

$$y(0) = 0 \Rightarrow c_1 = 0$$

$$y'(0) = -1$$

$$\Rightarrow \left[2c_2 e^{2t} \sin\sqrt{3} + c_2\sqrt{3}e^{2t}\cos\sqrt{3}t \right]_{t=0} = -1$$

$$\Rightarrow \quad c_2 = -\frac{1}{\sqrt{3}}$$

$$\therefore \quad y(t) = -\frac{1}{\sqrt{3}} e^{2t} \sin\sqrt{3}t$$

19) Two of the roots are $2-i$ and $2+i$. The other root is not specified, but must be real. The characteristic eqn could be

$$(r-(2-i))(r-(2+i))(r-2)$$

$$= r^3 - 6r^2 + 13r - 10$$

$$\Rightarrow y''' - 6y'' + 3y' - 10y = 0$$

32) The general solution is

$$y(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

Since $\beta \neq 0$, the solution will oscillate with amplitude determined by $e^{\alpha t}$.

For $\alpha < 0$, the amplitude decreases.

" $\alpha > 0$, " " increases

" $\alpha = 0$ " " remains constant

43) Characteristic Equation:

$$r^3 + 6r^2 + 12r + 8 = 0$$

$$\Rightarrow (r+2)^3 = 0$$

Each repeated root corresponds to a linearly independent solution with an increasing power of t.

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + c_3 t^2 e^{-2t}$$

50)

(a) The DE is given by

$$x'' + 2x' + 3x = 0 \quad ; \quad x(0) = 1, \quad x'(0) = 0$$

Char Eqn: $r^2 + 2r + 3 = 0 \implies r = -1 \pm i\sqrt{2}$

So, $x(t) = e^{-t}(c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t)$

$$x'(t) = e^{-t}\left((\sqrt{2}c_2 - c_1)\cos\sqrt{2}t - (\sqrt{2}c_1 + c_2)\sin\sqrt{2}t \right)$$

$x(0) = 1 \implies c_1 = 1$

$x'(0) = 0 \implies c_2 = 1/\sqrt{2}$

So, $x(t) = e^{-t}\left(\cos\sqrt{2}t + \frac{1}{\sqrt{2}} \sin\sqrt{2}t \right)$

$$x'(t) = e^{-t}\left(-\frac{3\sqrt{2}}{2}\sin\sqrt{2}t\right)$$

To find the maximum,

$x'(t) = 0 \implies t = \frac{\pi}{\sqrt{2}}$

$$\implies x_{max} = x\left(\frac{\pi}{\sqrt{2}}\right) = -e^{-\frac{\pi}{\sqrt{2}}}$$

(b) $x'' + 2x' + 10x = 0$

$r^2 + 2r + 10 = 0 \implies r = -1 \pm 3i$

$\implies x(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$

Using initial conditions,

$x(t) = \frac{2}{3} e^{-t} \sin 3t$

Setting $x'(t) = 0$ gives $\tan 3t = 3$

$\implies t = .416$ rad

$\implies x_{max} = .4172$

(c) $x'' + 4x' + 4 = 0$

$\Rightarrow (r+2)^2 = 0$

$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

From initial conditions,

$x(t) = 2te^{-2t}$

$x'(t) = 2e^{-2t} - 4te^{-2t}$

Setting $x'(t) = 0 \Rightarrow t = \frac{1}{2}$

so $x_{max} = \frac{1}{e}$

63)

(a) $x(t) = e^{-5t}\left(\cos\sqrt{39}t + \frac{5}{\sqrt{39}}\sin\sqrt{39}t\right)$

(underdamped)

(b) $x(t) = (1+8t)e^{-8t}$

(critically damped)

(c) $x(t) = \frac{1}{3}\left(4e^{-4t} - e^{-16t}\right)$

(overdamped)

§4.4 / 5, 6, 19, 30, 36, 46

5) $y'' - 2y' + 2y = 4$

look for a constant solution:

$$y_p = 2$$

6) $y'' - y = -2\cos t$

By inspection, $y_p = \cos t$

19) $y' + y = t$

by inspection, $y_p = t - 1$

For the homogeneous sol'n, the characteristic eqn is $r + 1 = 0 \Rightarrow y_h(t) = ce^{-t}$

$\therefore \quad y(t) = y_h + y_p = ce^{-t} + t - 1$

30) $y'' - y = t\sin t$

$$y_h = c_1 e^{t} + c_2 e^{-t}$$

The RHS implies a particular solution of the form

$$y_p = (At + B)\cos t + (Ct + D)\sin t$$

Differentiating and substituting we get,

$$y_p'' - y_p = -2Ct\sin t - 2At\cos t - (2A+2D)\sin t + (2C-2B)\cos t$$

$$\Rightarrow A = 0, \quad B = -\frac{1}{2}, \quad C = -\frac{1}{2}, \quad D = 0$$

$$y(t) = c_1 e^t + c_2 e^{-t} - \frac{1}{2}(t\sin t + \cos t)$$

36) $y'' + 3y' = \sin t + 2\cos t$

Homog: $r^2 + 3r = 0 \Rightarrow y_h(t) = c_1 + c_2 e^{-3t}$

Particular: $y_p = A\cos t + B\sin t$

$$\Rightarrow (-A+3B)\cos t + (-B-3A)\sin t = \sin t + 2\cos t$$

$$\Rightarrow A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

$$\therefore y(t) = c_1 + c_2 e^{-3t} + \frac{1}{2}(\cos t - \sin t)$$

46) $y'' + 9y = \cos 3t \qquad y(0) = 1, \quad y'(0) = -1$

homog: $r^2 + 9 = 0 \Rightarrow y_h = c_1\cos 3t + c_2 \sin 3t$

Particular: $y_p = t(A\cos 3t + B\sin 3t)$

$$\Rightarrow A = 0, \quad B = \frac{1}{6}$$

$$y_p = \frac{1}{6} t \sin 3t$$

$$\therefore y(t) = c_1\cos 3t + c_2\sin 3t + \frac{1}{6}t\sin 3t$$

$y(0) = 1 \Rightarrow c_1 = 1$

$y'(0) = -1 \Rightarrow c_2 = -\frac{1}{3}$

$$\boxed{y(t) = \cos 3t - \frac{1}{3}\sin 3t + \frac{1}{6}t\sin 3t}$$