7.2 3, 5, 7, 11, 20

3) $x' = x + y + 2xy$

$y' = -2x + y + y^3$

$\Rightarrow$ Find equilibria

$x' = 0$, $y' = 0$

- $x + \cancel{y} + 2xy = -2x + \cancel{y} + y^3$

$3x + 2xy = y^3$

$x(3+2y) = y^3$

$x = \dfrac{y^3}{3+2y}$

- $\dfrac{y^3}{3+2y} + y + \dfrac{2y^3}{3+2y}y = 0$

$y^3 + 3y + 2y^2 + 2y^4 = 0$

$y = -1$, $y = 0$

- $x = \dfrac{y^3}{3+2y} \rightarrow x = -1$, $x = 0$

Therefore $\boxed{\text{equilibria} = (0,0) \text{ and } (-1,-1)}$

(next page)

7.2 3, 5, 7, 11, 20

3 con't)

=> Compute Jacobian

$$J = \begin{bmatrix} 1+2y & 1+2x \\ -2 & 1+3y^2 \end{bmatrix}$$

=> Analyze type and stability @ (0,0)

$$J = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$(1-\lambda)^2 + 2 = 0$$

$$1 - 2\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 2\lambda + 3 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-12}}{2} = 1 \pm \frac{1}{2}\sqrt{2 \cdot 4 \cdot -1}$$

$\lambda_1, \lambda_2 = 1 \pm i\sqrt{2}$ therefore unstable repelling spiral @ (0,0)

(next page)

7.2 3, 5, 7, 11, 20

3 cont)

@ (-1, -1)

$$J = \begin{bmatrix} -1 & -1 \\ -2 & 4 \end{bmatrix}$$

$(-1-\lambda)(4-\lambda) - 2 = 0$

$-4 - 3\lambda + \lambda^2 - 2 = 0$

$\lambda^2 - 3\lambda - 6 = 0$

$$\lambda = \frac{3 \pm \sqrt{9+36}}{2} = \frac{3}{2} \pm \frac{1}{2}\sqrt{9 \cdot 5}$$

$$\boxed{\lambda_1, \lambda_2 = \frac{3}{2} \pm \frac{3}{2}\sqrt{5} \quad \text{therefore unstable sattle @ } (-1, -1)}$$

7.2 5, 7, 11, 20

5) $x' = x + y^2$

$y' = x^2 + y^2$

=> find equilibria

$x + \cancel{y^2} = x^2 + \cancel{y^2} \rightarrow x - x^2 = 0 \rightarrow x(1-x) = 0$

$x = 0, \; x = 1$

$x^2 + y^2 = 0 \rightarrow 0^2 + y^2 = 0 \rightarrow y = 0$

$x^2 + y^2 = 0 \rightarrow 1^2 + y^2 = 0 \rightarrow y = i$

$\boxed{\text{equilibrium @ } (0,0) \text{ and } \cancel{(1,i)}}$ uninteresting

don't penalize if they don't have this one

=> Find Jacobian

$$J = \begin{bmatrix} 1 & 2y \\ 2x & 2y \end{bmatrix}$$

=> Analyze type and stability:

@ $(0,0)$ $J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $(1-\lambda)(-\lambda) = 0$

$-\lambda + \lambda^2 = 0 \rightarrow \lambda(\lambda - 1) = 0$

$\boxed{\lambda_1 = 0 \quad \lambda_2 = 1 \quad \text{Repelling Node unstable @ } (0,0)}$

$\underline{7.2}$ 7, 11, 20

7) $x' = 1 - xy$
$y' = x - y^3$

=> Find equilibrium

$x = y^3 \rightarrow 0 = 1 - y^3 y \rightarrow 0 = 1 - y^4 \rightarrow y^4 = 1$

$y = \pm 1 \quad x = \pm 1$

$\boxed{\text{equilibria @ } (1,1) \text{ and } (-1,1)}$

=> Find Jacobian

$$J = \begin{bmatrix} -y & -x \\ 1 & -3y^2 \end{bmatrix}$$

=> analyze type and stability

@ $(1,1)$ $\quad J = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$ $\quad$ $(-1-\lambda)(-3-\lambda)+1=0$
$3 + 4\lambda + \lambda^2 + 1 = 0$
$\lambda^2 + 4\lambda + 4 = 0$

$$\lambda = \frac{-4 \pm \sqrt{16-16}}{2} = -2 \text{ multiple root}$$

$\boxed{\lambda_1 = \lambda_2 = -2 \quad \text{Asymptotically stable attracting star @ } (1,1)}$

(next page)

$\underline{7.2}$ 7, 11, 20

7 cont) @ $(-1,-1)$ $J = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$

$(1-\lambda)(-3-\lambda) - 1 = 0$

$-3 + 2\lambda + \lambda^2 - 1 = 0$

$\lambda^2 + 2\lambda - 4 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4+16}}{2} = -1 \pm \frac{\sqrt{4 \cdot 5}}{2}$$

$\lambda_1, \lambda_2 = -1 \pm \sqrt{5}$ therefore,

saddle unstable @ $(-1,-1)$

7.2 11, 20

11) $\ddot{x} + \dot{x} + x + x^3 = 0$

=> convert

$x' = y$

$y' = -x^3 - x - y$

=> locate equilibria

$y = 0$

$-x^3 - x = 0 \rightarrow x(-x^2 - 1) = 0$

$x = 0, \; x = i$ uninteresting

$$\boxed{\text{equilibrium @ } (0,0)}$$

=> Find Jacobian

$$J = \begin{bmatrix} 0 & 1 \\ -3x^2 - 1 & -1 \end{bmatrix}$$

=> analyze type and stability

@ $(0,0)$ $\quad J = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ $\quad (-\lambda)(-1-\lambda) + 1 = 0$

$\lambda + \lambda^2 + 1 = 0$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\boxed{\lambda_1, \lambda_2 = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \quad \text{attracting spiral @ } (0,0) \text{ Asymptotically stable}}$$

7.2 20

20) $\ddot{x} - \dot{x} + x = 0$

Intuition: anything goes, let's guess it will be unstable because of negative damping

=> convert

$x' = y$

$y' = -x + y$

=> Check $(0,0)$, find Jacobian

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \qquad \begin{aligned} &(-\lambda)(1-\lambda) + 1 = 0 \\ &-\lambda + \lambda^2 + 1 = 0 \to \lambda^2 - \lambda + 1 = 0 \end{aligned}$$

$$\lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$\lambda_1, \lambda_2 = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$ repelling spiral

unstable @ $(0,0)$