APPM 2360 Spring 2009
Solutions for Homework 3

Sec 1.4:

2.Calculator Again Consider the IVP $y' = ty$, $y(0) = 1$.

(a) *Use Euler's method to approximate the solution at* $t = 1$ *with step sizes* 1, 1/2, 1/4, 1/8.

(b) *Solve the problem exactly, and compare the reult at* $t = 1$ *with the approximations calculated in part (a).*

Solutions:

(a)

Euler's Method with Different Step Sizes							
$h = 1$		$h = 1/2$		$h = 1/4$		$h = 1/8$	
t	$y \approx$	t	$y \approx$	t	$y \approx$	t	$y \approx$
0	1	0	1	0	1	0	1
1	**1**	0.5	1	0.25	1	0.125	1
		1	**1.25**	0.50	1.062	0.250	1.0156
				0.75	1.195	0.375	1.0474
				1	**1.419**	0.50	1.0965
						0.625	1.1650
						0.750	1.2560
						0.875	1.3737
						1	**1.5240**

(b)Separating variables: $\frac{dy}{y} = tdt$, thus $\ln|y| = \frac{t^2}{2} + c \Rightarrow y = Ce^{t^2/2}$. Notice $y(0) = 1$ yeilds $C = 1 \Rightarrow y(t) = e^{t^2/2}$. Plug in $t = 1$, we have $y(1) = e^{1/2} \approx 1.6487$.
The estimate value of $y(1)$ with different stepsize h is listed in boldface as shown above, thus:
$h = 1$:, $Error = 1.6487 - 1 = 0.6487$;
$h = 1/2$:, $Error = 1.6487 - 1.25 = 0.3987$;
$h = 1/4$:, $Error = 1.6487 - 1.419 = 0.2297$;
$h = 1/8$:, $Error = 1.6487 - 1.524 = 0.1247$;

7. *Solve the problem below numerically using various step sizes. Compare with values of exact solutions when possible.* $y' = t - y$, $y(0) = 2$

Solutions:

With stepsize $h = 0.05$ and Euler's method we obtain the following results:(other stepsizes and numerical methods are also engouraged)

Euler's Method ((h = 0.05))			
t	$y \approx$	t	$y \approx$
0	2	0.6	1.2211
0.1	1.8075	0.7	1.1630
0.2	1.6435	0.8	1.1204
0.3	1.5053	0.9	1.0916
0.4	1.3903	**1**	**1.0755**
0.5	1.2962		

Since the DE is not separable, you don't have to do any comparison.

10. *Solve the problem below numerically using various step sizes. Compare with values of exact solutions when possible.* $y' = -ty$, $y(0) = 1$

Solutions:

Choose $h = 0.01$, still use Euler's method results is as follows:

Euler's Method ((h = 0.01))			
t	$y \approx$	t	$y \approx$
0	1	0.6	0.8375
0.1	0.9955	0.7	0.7850
0.2	0.9812	0.8	0.7284
0.3	0.9574	0.9	0.6692
0.4	0.9249	**1**	**0.6086**
0.5	0.8845		

It's easy to find the Analytical solution is $y(t) = e^{-t^2/2}$. So $y(1) = e^{-1/2} = 0.6065$. So, $Error = |0.6065 - 0.6086| = 0.0021$

12. Nasty Surprise Use Euler's method with $h = 0.25$ to approximate the solution of $y' = y^2$, $y(0) = 1$, at $t = 0.50$, $t = 0.75$, and $t = 1$. Verify that the exact solution is $y(t) = 1/(1-t)$; does this help explain what happened to the Euler approximations?

Solutions:

Choose $h = 0.25$, then the value of y at each step is as follows:

Euler's Method ((h = 0.25))		
t	$y \approx$	$y' = y^2$
0	1	1
0.25	1.25	1.5625
0.50	1.6406	2.6917
0.75	2.3135	5.3525
1	**3.6517**	

For analytical solution: $\frac{dy}{y^2} = dt \Rightarrow -y^{-1} = t + c \Rightarrow$. Plug in $y(0) = 1 \Rightarrow c = -1 \Rightarrow y(t) = \frac{1}{1-t}$. And we can see that at $t = 1$, $y = \infty$, which sugests Euler method doesn't work very well for this problem.

22. *Solving* $y' = -ty, y(0) = 1$ *using Runge-Kutta method.*

Solutions:

Using the 4th-order Runge Kutta method and $h = 0.01$, we obtain the following table:

Runge-Kutta Method ((h = 0.01))			
t	$y \approx$	t	$y \approx$
0	1	0.6	0.8353
0.1	0.9950	0.7	0.7827
0.2	0.9802	0.8	0.7261
0.3	0.9560	0.9	0.6670
0.4	0.9231	**1**	**0.6065**
0.5	0.8825		

We have already known in **problem (10)** that, the exat solution at 1 is $y(1) = 0.6065$. So the Runge-Kutta approximate the solution within given accuracy.

Sec 1.5:

2. $ty' + y = 2, \quad y(0) = 1$

Solutions:

From the DE, we have $y' = \frac{2-y}{t}$, so $f(t,y) = \frac{2-y}{t}$, and the continous region for f is:

$$R = \{(t,y) | t \in (-\infty, 0) \cup (0, \infty), y \in \mathbb{R}\}$$

while the initial value $(t_0, y_0) = (0, 1) \notin R$. Therefore, the answer to (a) is "no"–Picard's Thm doesn't apply to the IVP. And for (c), the answer is "yes", such as $(1, 5)$. The set of such points are R as given above.

7. $y' = \ln|y-1|, \quad y(0) = 2$

Solutions:

Since $f(t,y) = \ln|y-1|$, and $f_y = \frac{1}{y-1}$(Seperate the cases when $y > 1$ and $y < 1$, break the absolute value and you'll get it.) So the continuous region for f and f_y are the same, *i.e.*

$$R = \{(t,y) | \mathbb{R} \times (-\infty, 1) \cup (1, \infty)\}$$

and this is the largest region for Picard's Thm to hold. Obviously, the initial value $(t_0, y_0) = (0, 2) \in R$, so there exists a unique solution passing through $(0, 2)$.

19. $y' = y^2, \quad y(0) = 1$

Solutions:

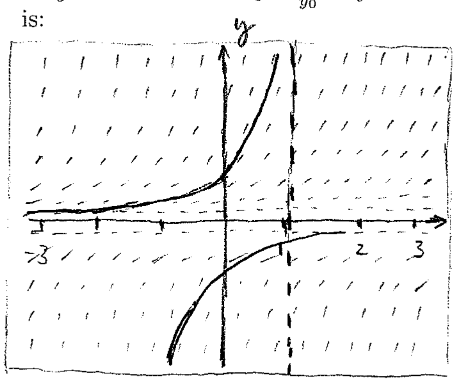
(a).Since $f(t,y) = y^2$ and $f_y = 2y$, the two functions are continuous everywhere in the $t - y$ plane, that is Picard's condition holds everywhere, thus the largest region is just $\mathbb{R} \times \mathbb{R}$.

(b). (c). By the results from **Sec 1.4 Problem 12**, we have $y(t) = \frac{1}{1-t}$. As when $t = 1$, $y = \infty$ and also according to the graph, the largest interval the the solution is defined is: $(-\infty, 1)$.

(d). Seperate the variables and integrate we have: $-y^{-1} = t + c$, plug in $y(t_0) = y_0$ we obtain: $-1/y_0 = t_0 + c \Rightarrow c = -t_0 - \frac{1}{y_0}$ so

$$y(t) = \frac{1}{t_0 + \frac{1}{y_0} - t}$$

So $y = \infty$ when $t = t_0 + \frac{1}{y_0}$. By observing the graph, this means: the largest interval that the solution exists is:



$$(-\infty, t_0 + \frac{1}{y_0}), \quad \text{if} \quad y_0 > 0;$$

$$(t_0 + \frac{1}{y_0}, \infty), \quad \text{if} \quad y_0 < 0.$$

20. Nonuniqueness Show that IVP $y' = y^{1/3}$, $y(0) = 0$ exhibits nonunique solutions and sketch graphs of several possibilities. What does Picard's Thm tell you for the problem?

Solutions:
Because $f = y^{1/3}$ is continuous everywhere, so by Picard's Thm, a solution exists for any initial value. However, $f_y = 1/3y^{-2/3}$ is not continuous for $y = 0$. So Picard's Thm doesn't guarantee the uniqueness for points where $y = 0$. Seperate variables, we get $y^{-1/3}dy = dt$ thus,

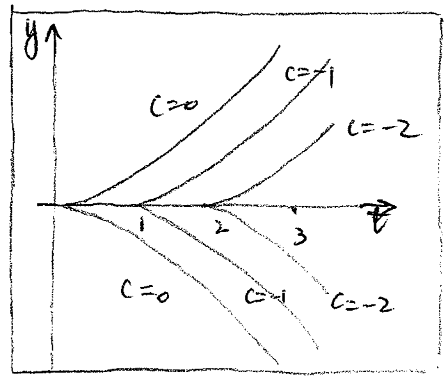
$$\frac{3}{2}y^{2/3} = t + c.$$

Plug in $y(0) = 0 \Rightarrow c = 0$. So we can get one solution:

$$y(t) = \pm \left(\frac{2}{3}\right)^{3/2} |t|^{3/2}$$

Notice that $y = 0$ is also a solution. So, combine both of them we have an infinite number of solutions as follows:

$$y(t) = \begin{cases} 0 & t < |c| \\ \pm \left(\frac{2}{3}\right)^{3/2} (t+c)^{3/2} & t \geq |c| \geq 0 \end{cases}$$



Sec 2.1:

4. Solutions: Second-order, linear, nonhomogeneous, variable coefficients.
8. Solutions: Second-order, nonlinear.

13. $L(y) = y' + y^2$
Solutions:
Nonlinear, since $L(ky) = ky' + k^2y \neq kL(y) = ky' + ky^2$, vialates (3).

29. Solutions: Easy to verify y_1 and y_2 are solutions. For $y = c_1e^{2t} + c_2e^{3t}$,
$y'' - 5y' - 6y = c_14e^{2t} + c_29e^{3t} - 5(c_12e^{2t} + c_23e^{3t}) + 6(c_1e^{2t} + c_2e^{3t}) = 0$

44. Solutions:
(1),just verify y_p is the solution;
(2), Since $L(y) = y' + 2y$ is linear (verify it), for the nonhomogeneous problem:$y' + 2y = 0$, seperate variavles, we get: $y(t) = ce^{-2t}$, so the solution for the original problem is $y(t) = y_h + y_p = ce^{-2t} + 4\sin t - 2\cos t$.