2.2

2. $\frac{dy}{dt}+2y=3e^{t}$ Euler-Lagrange Method. First note $p(t)=2$.

$y_h = ce^{-\int 2dt}$

$= ce^{-2t}$.

Now solve

$v'(t)e^{-2t}=3e^{t}$

$\Rightarrow v'(t)=(3e^{t})e^{2t} = 3e^{3t}$

Thus $v(t)=\int 3e^{3t}dt = e^{3t}$.

Thus $y_p = e^{3t}e^{-2t}=e^{t}$.

Thus $y(t)=y_h+y_p = ce^{-2t}+e^{t}$.

10. $\cos t \frac{dy}{dt}+y\sin t = 1$.

$\Rightarrow \frac{dy}{dt}+y\tan t = \sec t$. Euler-Lagrange Method. Note $p(t)=\tan(t)$

$y_h = ce^{-\int \tan(t)dt} = ce^{-\ln|\sec(t)|} = ce^{\ln\frac{1}{|\sec(t)|}} = c\frac{1}{|\sec(t)|} = \tilde{c}\frac{1}{\sec(t)}$

Now solve

$v'(t)\frac{1}{\sec(t)} = \sec t$

$\Rightarrow v'(t)=\sec^2(t)$

$\Rightarrow v(t)=\int \sec^2(t)dt = \tan(t)+C$,

So $y_p = v(t)e^{-\int p(t)dt} = \frac{\tan(t)}{\sec(t)} = \frac{\sin(t)}{\cancel{\cos(t)}}\cdot\cancel{\cos(t)} = \sin(t)$.

So $y=y_p+y_h = \sin(t)+\tilde{c}\frac{1}{\sec(t)}$.

18. $\frac{dy}{dt} - \frac{3}{t}y = t^3$. Note $p(t) = \frac{-3}{t} \Rightarrow e^{\int p(t)} = e^{-3\ln|t|+c} = k|t^{-3}| = \frac{k}{t^3}$

Integrating factor method

$t^{-3}\left(\frac{dy}{dt} - \frac{3}{t}y\right) = \frac{t^3}{t^3}$

$\Rightarrow \frac{d}{dt}[t^{-3}y] = 1$

$\Rightarrow t^{-3}y = \int 1\, dt$

$\Rightarrow t^{-3}y = t + c$

$\Rightarrow y = t^4 + ct^3$.

IVP $y(1) = 4$

$\Rightarrow 4 = 1^4 + c1^3$

$\Rightarrow 4 = 1 + c$

$\Rightarrow c = 3$.

Thus $y(t) = t^4 + 3t^3$.

26. $y' + y = \frac{1}{1+e^t}$. Note $p(t) = 1 \Rightarrow e^{\int p(t)dt} = e^{\int 1 dt} = e^{t+c} = ke^t$.

Integrating factor

$e^t[y'+y] = \frac{e^t}{1+e^t}$

$\frac{d}{dt}[e^t y] = \frac{e^t}{1+e^t}$

$e^t y = \int \frac{e^t}{1+e^t} dt$ $[u = 1+e^t, du = e^t dt]$

$\Rightarrow e^t y = \int \frac{1}{u} du$

$\Rightarrow e^t y = \ln|u| + c$

$\Rightarrow y = e^{-t}\ln|1+e^t| + ce^{-t}$

$\Rightarrow y = e^{-t}\ln(1+e^t) + ce^{-t}$ $[1+e^t > 0$ always$]$

2.2 | 34. $\frac{dy}{dt} + p(t)y = q(t)y^{\alpha}$

a) $y^{-\alpha}\left(\frac{dy}{dt} + p(t)y\right) = y^{-\alpha}(q(t)y^{\alpha})$

$\Rightarrow \frac{dy}{dt}y^{-\alpha} + p(t)y^{-\alpha+1} = q(t)$, (*)

Let $v = y^{1-\alpha}$ $\quad \frac{dv}{dt} = (1-\alpha)y^{1-\alpha-1}\frac{dy}{dt}$ (chain rule)

$\Rightarrow v = y^{-\alpha+1}$ $\quad \Rightarrow \frac{dv}{dt} = (1-\alpha)y^{-\alpha}\frac{dy}{dt}$.

$\Rightarrow \frac{dv}{dt}\frac{1}{(1-\alpha)} = y^{-\alpha}\frac{dy}{dt}$

$\Rightarrow$ (*) is equivalent to

$\frac{dv}{dt}\frac{1}{(1-\alpha)} + p(t)v = q(t)$

$\Rightarrow \frac{dv}{dt} + (1-\alpha)p(t)v = q(t)(1-\alpha)$, which is linear in v, is equivalent to (*).

b) $y' - y = y^3$ (**)

a) with $\alpha = 3$ implies (**) is equivalent to

$\frac{dv}{dt} + (1-3)(-v) = (1-3)$ $\quad$ where $v = y^{-3+1}$.

Using integrating factor, $p(t) = (1-3) \Rightarrow e^{\int p(t)dt} = e^{-2t}$.

Thus $e^{-2t}\left[\frac{dv}{dt} + 2v\right] = -2e^{-2t}$

$\Rightarrow \frac{d}{dt}\left[e^{-2t}v\right] = -2e^{-2t}$

$\Rightarrow e^{-2t}v = \int -2e^{-2t}dt$

$\Rightarrow e^{-2t}v = e^{-2t} + c$

$\Rightarrow v = e^{2t}(e^{-2t} + c)$

$\Rightarrow v = 1 + ce^{2t}$

$\Rightarrow y^{-2} = 1 + ce^{2t}$ (as $v = y^{-2}$ by definition)

$\Rightarrow y^2 = (1 + ce^{2t})^{-1}$

$\Rightarrow y = \pm\sqrt{\frac{1}{1+ce^{2t}}}$.

c) When $\alpha=0$ or $\alpha=1$, no substitution is needed to reduce the differential equation to a first order linear system,

$$\frac{dy}{dt}+p(t)\,y=q(t)\,y^0 \Rightarrow \frac{dy}{dt}+p(t)y=q(t) \checkmark$$

$$\frac{dy}{dt}+p(t)y=q(t)y \Rightarrow \frac{dy}{dt}+[p(t)-q(t)]y=0$$

$$\Rightarrow \frac{dy}{dt}+\tilde{p}(t)y=0 \quad [\tilde{p}(t) \overset{def}{:=} p(t)-q(t)]$$

$\checkmark$

Integrating factor, or Variation of Parameters can be used on these linear systems,

$\boxed{2.3}$ 4. $y(0)=100=y_0$. Basic Model $y(t)=y_0e^{-kt}$.

$y(1)=75$.

$$\Rightarrow y(1)=y_0e^{-k(1)} \Rightarrow 75=100e^{-k} \Rightarrow \frac{3}{4}=e^{-k} \Rightarrow \ln(\tfrac{3}{4})=-k$$

$$\Rightarrow -\ln(\tfrac{3}{4})=k \Rightarrow \ln(\tfrac{4}{3})=k.$$

Thus $y(t)=100e^{-\ln(\frac{4}{3})t} \Rightarrow y(t)=100\,e^{\ln((\frac{4}{3})^{-t})}$

$$\Rightarrow y(t)=100\left(\tfrac{4}{3}\right)^{-t}.$$

To find the halflife we solve $y(t)=\frac{y_0}{2}$ for t and get

$$\frac{y_0}{2}=100(\tfrac{4}{3})^{-t} \Rightarrow 50=100(\tfrac{4}{3})^{-t}$$

$$\Rightarrow \tfrac{1}{2}=(\tfrac{4}{3})^{-t}$$

$$\Rightarrow \ln(\tfrac{1}{2})=\ln\left((\tfrac{4}{3})^{-t}\right)$$

$$\Rightarrow \ln(\tfrac{1}{2})=-t\ln(\tfrac{4}{3})$$

$$\Rightarrow \frac{\ln(\frac{1}{2})}{-\ln(\frac{4}{3})}=t$$

$$\Rightarrow \frac{\ln(\frac{1}{2})}{\ln(\frac{3}{4})}=t \Rightarrow \frac{-\ln(2)}{-\ln(\frac{4}{3})}=\frac{\ln(2)}{\ln(\frac{4}{3})}=t.$$

5. Basic Model: $y(t) = y_0 e^{-kt}$.

we are given $\frac{y_0}{2} = y_0 e^{-k5}$

$\Rightarrow \frac{1}{2} = e^{-5k} \Rightarrow -5k = \ln(\frac{1}{2})$

$\Rightarrow k = -\frac{1}{5}\ln(\frac{1}{2})$

$\Rightarrow k = \frac{1}{5}\ln(2)$.

Thus $y(t) = y_0 e^{-\frac{1}{5}\ln(2)t}$.

we want to solve $y(t) = \frac{y_0}{10}$ for t.

$\frac{y_0}{10} = y_0 e^{-\frac{1}{5}\ln(2)t}$

$\Rightarrow \frac{1}{10} = e^{-\frac{1}{5}\ln(2)t}$

$\Rightarrow \ln(\frac{1}{10}) = -\frac{1}{5}\ln(2)t$

$\Rightarrow -\ln(10) = -\frac{1}{5}\ln(2)t$

$\Rightarrow \frac{5\ln(10)}{\ln(2)} = t$.

11. Basic Model $y(t) = y_0 e^{-kt}$,

we know $y(t) = \frac{y_0}{2} \Rightarrow t = 258$.

Thus $\frac{y_0}{2} = y_0 e^{-k258}$

$\Rightarrow \frac{1}{2} = e^{-k258}$

$\Rightarrow \ln(\frac{1}{2}) = -k258$

$\Rightarrow k = \frac{-\ln(\frac{1}{2})}{258}$

$\Rightarrow k = \frac{\ln(2)}{258}$.

we want to solve $y(t) = 0.05y_0$ for t, So, $0{,}05 y_0 = y_0 e^{-\frac{\ln(2)}{258} t}$

$\Rightarrow 0{,}05 = e^{-\frac{\ln(2)}{258} t}$

$\Rightarrow \ln(0{,}05) = -\frac{\ln(2)}{258} t$

$\Rightarrow \frac{-\ln(0{,}05) \cdot 258}{\ln(2)} = t$

$\Rightarrow \frac{\ln(20) 258}{\ln(2)} = t$

13. Basic Model $y(t) = y_0 e^{-kt}$.

we are given $k = 0.1$, $y_0 = 0.002$

a) we want to find $y(t)$

$y(t) = 0.002 e^{-0.1t}$ [t in hours, y(t) in % bac]

b) Want to solve $y(t) = 0.001$ for t,

$0{,}001 = 0.002 e^{-0.1t}$

$\Rightarrow \frac{1}{2} = e^{-0.1t}$

$\Rightarrow \ln(\frac{1}{2}) = -0.1t$

$\Rightarrow -10 \ln(\frac{1}{2}) = t$

$\Rightarrow 10 \ln(2) = t$.

So $10 \ln(2)$ hours, assuming you are in a position of fortuitous circumstance where a DUI does not result in at least a temporary revocation of your license,

28. Model $y(t) = y_0 e^{kt}$,

we are given $y(0) = y_0 = 1000$, and $y(50) = 18000$.

we want to solve $y(50) = y_0 e^{k50}$ for k

$$18000 = 1000 e^{50k}$$

$$\Rightarrow 18 = e^{50k}$$

$$\Rightarrow \ln(18) = 50k$$

$$\Rightarrow \frac{\ln(18)}{50} = k.$$

So an in interest rate of $\frac{\ln(18)}{500}(100)\% = \frac{\ln(18)}{5}\%$

would produce this result.

30. Let's recover the model from the answer, just to be coy.

We have $A(t) = 0 \Rightarrow t = \frac{1}{r}\ln\left(\frac{d}{d - rA_0}\right)$

$$\Rightarrow rt = \ln\left(\frac{d}{d - rA_0}\right)$$

$$\Rightarrow e^{rt} = \frac{d}{d - rA_0}$$

$$\Rightarrow \frac{d - rA_0}{d} e^{rt} = 1$$

$$\Rightarrow e^{rt} - \frac{rA_0}{d} e^{rt} - 1 = 0$$

$$\Rightarrow de^{rt} - rA_0 e^{rt} - d = 0$$

$$\Rightarrow rA_0 e^{rt} - de^{rt} + d = 0$$

$$\Rightarrow rA_0 e^{rt} - de^{rt} + d = A(t) \quad (\text{as } A(t) = 0 \text{ for this } t)$$

$$\Rightarrow \underbrace{(rA_0 - d)}_{\substack{\text{'growth' per year} \\ \text{(actually negative)}}} e^{rt} + \underbrace{d}_{\text{correction for withdrawl at time zero}} = A(t)$$

If $d \leq rA_0$ then $A(t) \geq d$ for all t (as $rA_0 - d \geq 0$), and the account is never depleted. This can also be seen as if $d \leq rA_0$ then $\frac{d}{d - rA_0} \leq 0$ which implies $\ln\left(\frac{d}{d - rA_0}\right)$ is not defined.