Homework 4 Solution Set

2.4) 4,7,15,18
2.5) 5,15,16
2.6) 4,13,20

Note: the ∴ symbol means "therefore" and "log" always means the natural log

Section 2.4

2.4.4)

Let $s =$ the amount of salt in the tank, so that $s(0) = 5$ lbs. $\frac{ds}{dt} = \text{Rate In} - \text{Rate Out} = 2\,\frac{\text{lbs}}{\text{gal}} \times 3\,\frac{\text{gal}}{\text{min}} - \frac{s}{20}\,\frac{\text{lbs}}{\text{gal}} \times 3\,\frac{\text{gal}}{\text{min}} = 6\,\frac{\text{lbs}}{\text{min}} - \frac{3}{20}\,s\,\frac{\text{lbs}}{\text{min}}$ so we end up with $\frac{ds}{dt} = 6 - \frac{3}{20}s \rightarrow$

$\frac{ds}{dt} + \frac{3}{20}s = 6 \rightarrow s = s_h + s_p = C e^{-\frac{3}{20}t} + \frac{20}{3}6$. To get C use the initial condition $s(0) = C + 40 = 5 \therefore C = -35$.

So our solution is $s(t) = -35 e^{-\frac{3}{20}t} + 40$ and will equal 25 when $-35 e^{-\frac{3}{20}t} = -5 \rightarrow e^{-\frac{3}{20}t} = \frac{1}{7} \rightarrow -\frac{3}{20}t = \text{Log}[1/7] \rightarrow t = \frac{20}{3}\text{Log}[7] \approx 12.97\,\text{min}$

2.4.7)

a) let p be the percentage concentration of pollutant in the lake, then the IVP is $\frac{dp}{dt} = \text{Rate In} - \text{Rate Out} = 0.01 \times \frac{40}{100}\,\frac{\%}{\text{year}} - p \times \frac{40}{100}\,\frac{\%}{\text{year}} \rightarrow \frac{dp}{dt} = 0.004 - 0.4\,p$ or equivilently $\frac{dp}{dt} + 0.4\,p = 0.004$ with $p(0) = 0.05\,\%$

b) $p = p_h + p_p = C e^{-0.4t} + \frac{0.004}{0.4} = C e^{-0.4t} + 0.01$. Now solve for C using the initial condition $C + 0.01 = 0.05 \therefore C = 0.04$ and we have $p(t) = 0.04 e^{-0.4t} + 0.01$

c) $0.04 e^{-0.4t} + 0.01 = 0.02 \rightarrow 0.04 e^{-0.4t} = 0.01 \rightarrow e^{-0.4t} = \frac{1}{4} \rightarrow t = -\frac{1}{0.4}\text{Log}[1/4] = \frac{5}{2}\text{Log}[4] \approx 3.47$ years

2.4.15)

Let T be the temperature of your house. $T' = k(95 - T)\,°\text{F} \rightarrow k = \frac{1}{4}\,\frac{1}{\text{hour}} \rightarrow T' = \frac{1}{4}(95 - T)\,\frac{°\text{F}}{\text{hour}}$ or equivilently

$T' + \frac{1}{4}T = \frac{95}{4}$and $T(0) = 75\,°\text{F}$. Now solve the ode describing the temperature of your house. $T = T_h + T_p = C e^{-\frac{1}{4}t} + 95$ with $C = -20$ to satisfy the initial condition. so we have $T(t) = -20 e^{-\frac{1}{4}t} + 95$

a) To get the time at 2pm just note that it is 2 hours from our initial time so we just look at $T(2) = -20 e^{-\frac{1}{2}} + 95 \approx 82.87\,°\text{F}$

b) Look for t such that $-20 e^{-\frac{1}{4}t} + 95 = 80 \rightarrow -20 e^{-\frac{1}{4}t} = -15 \rightarrow e^{-\frac{1}{4}t} = \frac{3}{4} \rightarrow t = 4\,\text{Log}[\frac{4}{3}] \approx 1.15$ hours or 1 hour and 9 minutes.

2.4.18)

our IVP is $T' = k(70 - T)$ or $T' + kT = k70$ with $T(0) = 35$ and $T(10) = 40$. The general solution for T is $T = C e^{-kt} + 70$. We need the two "initial" conditions because we know neither C nor k. Using the initial conditions we have the following pair of equations

$35 = C + 70$
$40 = C e^{-k*10} + 70$

The first gives us $C = -35$ and then the second becomes $-35 e^{-k*10} + 70 = 40 \to e^{-k*10} = 30/35 \to k = -\frac{1}{10}\text{Log}[30/35] = \frac{1}{10}\text{Log}[7/6]$ so our solution is

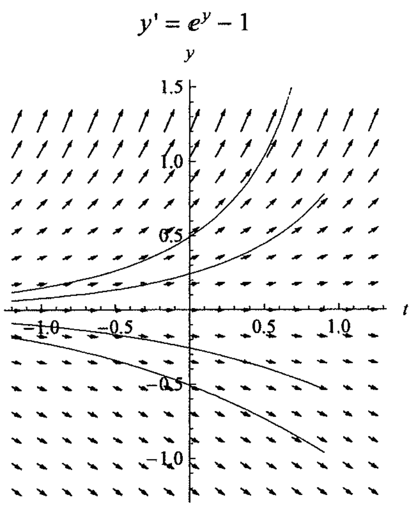
$$T(t) = -35 e^{-\frac{1}{10}\text{Log}[7/6]\, t} + 70$$

The Temperature at $t = 20$ is $T(t) = -35 e^{-2\,\text{Log}[7/6]} + 70 = 35\left(-\left(\frac{6}{7}\right)^2 + 2\right) = \frac{310}{7}$

<u>**Section 2.5**</u>

2.5.5)

$y' = e^y - 1$ has an equilibrium solution at $y = 0$ which we see is unstable since the sign of y' is positive when y is positive and negative when y is negative.



Note: FYI the code to generate a direction field using *Mathematica* is

```
Needs["VectorFieldPlots`"]
VectorFieldPlot[{1, y' equation}, {t, -a, a}, {y, -a, a}, Axes → True, AxesLabel → {t, y}]
```

where "**a**" is a number determining how big you want your box and "**y' equation**" is the equation for y'

2.5.15)

Let p be the population of the sample. $p(0) = 1000$, $p(1) = 2{,}000$, $p(\infty) = 100{,}000$ where $p(\infty)$ represents the value of the population at some very large time. If we assume a logistic growth model then we have that p is given by

$$p(t) = \frac{L}{1+\left(\frac{L}{p_0}-1\right)e^{-r t}}$$

The three conditions we have give us the following three equations to solve for L, r, and p_0

$$p_0 \equiv p(0) = 1000$$

$$\frac{L}{1+\left(\frac{L}{p_0}-1\right)e^{-r}} = 2000$$

$$\frac{L}{1+0} = 100{,}000$$

We can immediately read off $p_0 = 1000$ and $L = 100{,}000$ and if we substitute these into the middle equation we get

$$\frac{100{,}000}{1+(100-1)e^{-r}} = 2000 \to 50 = 1 + (100-1)e^{-r} \to \frac{49}{99} = e^{-r} \to r = \text{Log}\left[\frac{99}{49}\right]$$

So

$$p(t) = \frac{100{,}000}{1+(99)\left(\frac{49}{99}\right)^t}$$

a) $p(5) \approx 25{,}377$ (just plug 5 in for t in the above formula)

b) So we want to solve for t so that $\frac{100{,}000}{1+(99)\left(\frac{49}{99}\right)^t} = 50{,}000 \to \frac{100{,}000}{50{,}000} = 1 + (99)\left(\frac{49}{99}\right)^t \to 1 = (99)\left(\frac{49}{99}\right)^t \to$

$$\frac{1}{99} = \left(\frac{49}{99}\right)^t \to t = \frac{\text{Log}[1/99]}{\text{Log}[49/99]} \approx 6.534$$

2.5.16)

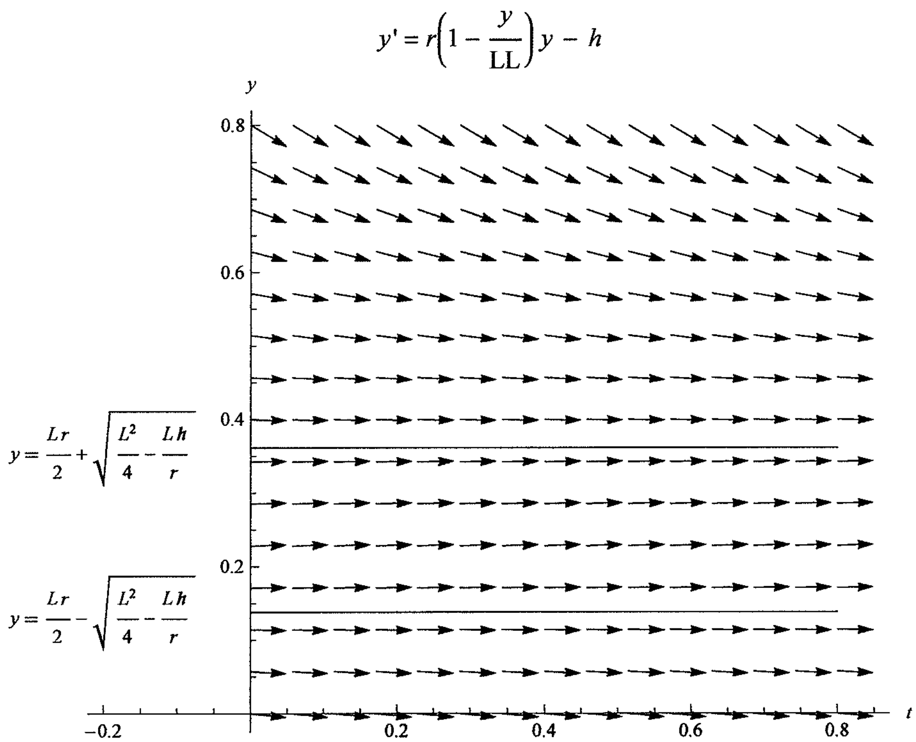
a) The IVP is $y' = r\left(1-\frac{y}{L}\right)y - h(t)$; $y(0) = y_0$, so now look for equilibrium points

$0 = r\left(1-\frac{y}{L}\right)y - h \to -\frac{r}{L}y^2 + r y - h = 0 \to y^2 - L y + \frac{L}{r}h = 0$ which has the y solutions of $y = \frac{rL}{2} \pm \sqrt{\left(\frac{L}{2}\right)^2 - \frac{L}{r}h}$.

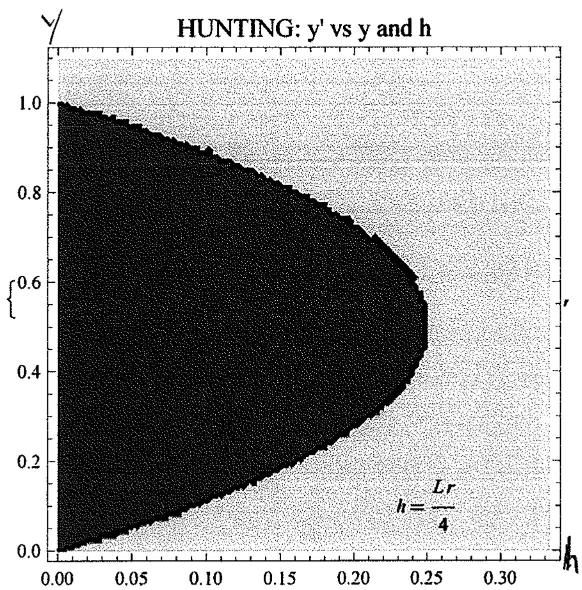
This tells us that the largest we can make h and still have a (real) equilibrium solution will be for $h_{\max} = r L/4$. It doesn't tell us *anything* else though, particularly whether or not such a situation is sustainable.

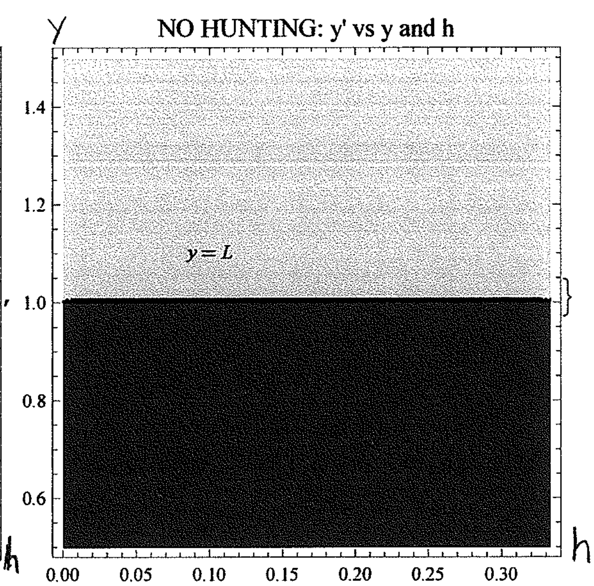
To get that this is the largest sustainable population note that if we write $y' = f(y) = r\left(1-\frac{y}{L}\right)y - h$ then if we look for the maximum value of $f(y)$ we first look at $f_y(y) = r - 2\frac{r}{L}y$ which tells us we have a maximum at $y = \frac{L}{2}$. So the maximum value of y' is $f_{max} = r\left(1 - \frac{1}{2}\right)\frac{L}{2} - h = \frac{rL}{4} - h$. This tells us any value of h greater than $\frac{rL}{4}$ will lead to a solution that is strictly decreasing.

b) The plot shows us the two equilibrium solutions we obtained in (a) along with the direction field and phase line. We see that so long as the population is above the the lower equilibrium solution, then the population will not die out. If however it drops below that the population will continue to decrease to zero. In the case of no hunting we have only one positive equilibrium value which is stable so we do not have the same problems in that case.



Something I find incredibly mysterious about this problem is that what we found in part (a) was a bifurcation point and yet the problem doesn't ask for a bifurcation diagram which makes very little sense to me. I'll include them because they make what's going on here a thousand times clearer than the above plot does.





Dark region means $y' > 0$ light region means $y' < 0$. Note that up to the point $h = r\,L/4$ for the Hunting Plot, that there are 2 equilibrium solutions, one stable and one unstable. Past the point $h = r\,L/4$ all solutions decrease regardless of their y value. However, note that in the No Hunting plot, a stable equilibrium *always* exists.

Section 2.6

2.6.4

$\frac{dx}{dt} = x + y$

$\frac{dy}{dt} = 2x + 2y$

a)

Nullclines: Just look for where x' and y' are zero.

$x' = 0 \to y = -x$

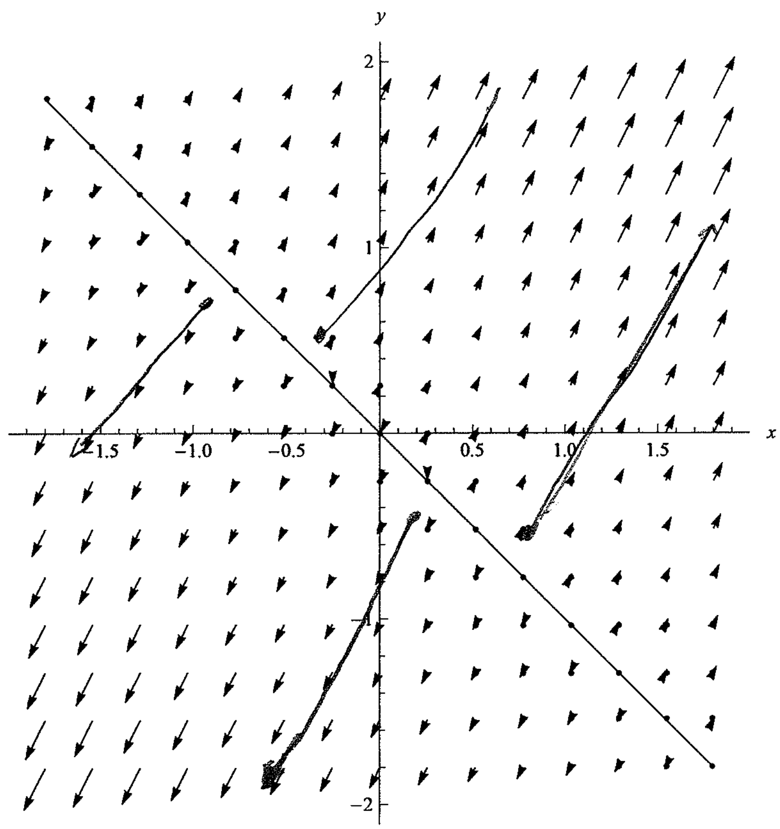
$y' = 0 \to y = -x$

Equilibrium points are where the nullclines intersect, which are at all points along the line $y = -x$

plot from a as well as b, & c are taken care of in the plot below

System Plot

Out[125]=



d) The only set of equilibria is the line $y = -x$ which we can see is unstable from both the graph and the following. If $x < -y$ then both $x'$and y' will always be negative and the solutions will shoot off to $-\infty$. Similarly if $x > -y$ then both $x'$and y' will always be positive and the solutions will shoot off to ∞.

2.6.13)

I'm just going to rewrite the problem here and add in with itallics what should be going through your head as you read it.

We have a population of rabits (x) and foxes (y). In the absence of foxes, the rabits obey the logistic population law .
(okay sweet, so our rabbit equation is going to be something like $x' = a_r\left(1 - \frac{x}{L}\right)x$*).*

The foxes will eat the rabbits but will die from starvtion if rabbits are not present *(oh noes, poor bunnies! ok well this sounds just like the Lotka-Volterra Equations on page 106 save that the R term in the first one becomes* $a_r\left(1 - \frac{x}{L}\right)x$*. This gives us* $x' = a_r\left(1 - \frac{x}{L}\right)x - c_r\,x\,y$ *and* $y' = -a_f\,y + c_f\,x\,y$ *where those c's and a's are just positive consants).*

However in this environment, hunters shoot rabbits but not foxes. *(man, those rabbits just can't catch a break. Anyhoo, the only modification we need to do is put in some term for the hunter/rabbit interaction. There's some flexiblility here and it would be reasonable to just assume it is some constant or something proportional to the number of rabbits. Either way you want to tweak your x' equation to be either* $x' = a_r\left(1 - \frac{x}{L}\right)x - c_r\,x\,y - H$ *or* $x' = a_r\left(1 - \frac{x}{L}\right)x - c_r\,x\,y - H\,x$ *where H is some positive constant.*

Final answer:
$y' = -a_f\,y + c_f\,x\,y$
and
$x' = a_r\left(1 - \frac{x}{L}\right)x - c_r\,x\,y - H$
or
$x' = a_r\left(1 - \frac{x}{L}\right)x - c_r\,x\,y - H\,x$

2.6.20

a)
Nullclines:
$x' = 0 \to x(2 - x - 2\,y) = 0 \to 2\,y\,x = 2\,x - x^2 \therefore y = 1 - \frac{1}{2}\,x$

$y' = 0 \to y(2 - 2\,x - y) = 0 \to 2\,x\,y = 2\,y - y^2 \therefore x = 1 - \frac{1}{2}\,y$ or $y = -2\,(x - 1)$

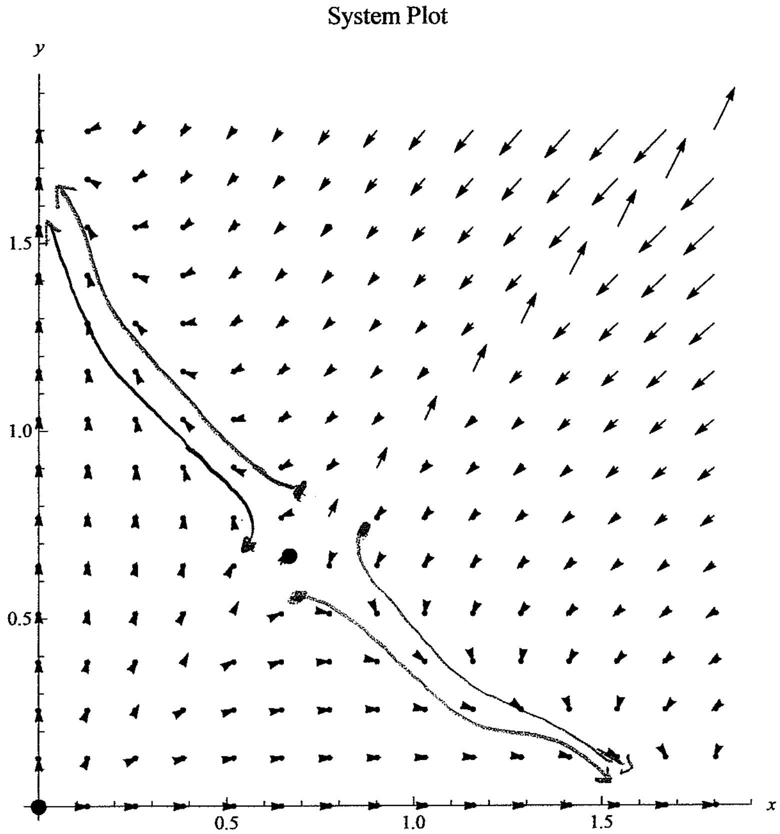
Equilibria where nullclines cross, so where $-2\,(x - 1) = 1 - \frac{1}{2}\,x$ which gives the point $\left\{\frac{2}{3}, \frac{2}{3}\right\}$, and we see that $\{0, 0\}$ is one as well.

b)
Both of these guys are unstable. Although, "look at the vector fields" is a good enough answer in my book, here's a more mathy explanation.

The $\{0, 0\}$ guy is easy to see because any perturbation away from it causes x' and y' to share the same sign as x and y respectively (so if x is positive, x' is positive which causes x to increase further). Just use the same trick on the system at the point $\left\{\frac{2}{3}, \frac{2}{3}\right\}$. Now we look at $x' = \left(x - \frac{2}{3}\right)\left(2 - \left(x - \frac{2}{3}\right) - 2\left(y - \frac{2}{3}\right)\right)$ so we can think about what is going on with x' if we move x a little bit away from $x = \frac{2}{3}$. We see that if $\{x, y\}$ is close to $\left\{\frac{2}{3}, \frac{2}{3}\right\}$ then if we make x a little bit bigger, x'will be positive and carry our trajectory further away so we know it will repel at least some nearby solutions.

c) Note that the lines $2 - x - 2\,y = 0$ and $2 - 2\,x - y = 0$ are what determine the sign changes in x' and y' respectively (the x and y out front in our Diff Eq system don't matter because populations have to be positive). This is what is carving up our plot into the quadrants that you see forming below with the arrows facing different directions.

Out[191]=



d) As we can see the species cannot coexist because any solution regardless of initial position will run into one of the axes (Note: they can coexist if the initial condition lies on the line $y = x$, but don't worry if you don't see where that comes from because you'll run into it in the linear algebra section and it isn't stable anyways). For coexistence to be possible then solutions must form closed loops.