$\boxed{\text{Sec 3.5: 13, 15, 16, 17, 20, 64}}$

13] Let $\Omega = \{(x,y) \in \mathbb{R}^2$ such that $x \geq y\}$

Ω is not a vector space because it's not closed under scalar multiplication:

Counter Ex] $(1,0) \in \Omega$ but $(-1)\cdot(1,0) = (-1,0) \notin \Omega$

15] Let $\Omega = \{$The set of all polynomials of even degree$\}$

Ω is not a vector space because it's not closed under addition.

Counter Ex] $p_1(x) = x^2 + x$, $p_2(x) = -x^2 + x$ $\Rightarrow p_1, p_2 \in \Omega$

$$p_1(x) + p_2(x) = 2x \Rightarrow p_1 + p_2 \notin \Omega$$

16] Let $\Omega = \{$The set of all diagonal 2×2 matrices$\}$

Ω is a vector space.

Proof] Let $M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in \Omega$ and $N = \begin{bmatrix} n_1 & 0 \\ 0 & n_2 \end{bmatrix} \in \Omega$

Then $\alpha M + \beta N = \begin{bmatrix} (\alpha m_1 + \beta n_1) & 0 \\ 0 & (\alpha m_2 + \beta n_2) \end{bmatrix} \in \Omega$

So Ω is closed under addition and scalar multiplication and Ω is a subset of $\mathbb{M}_{22}$, the set of all 2×2 matrices, so by the Vector Subspace Theorem Ω is a vector space.

17] $\Omega = \{M \in \mathbb{M}_{22}$ such that $|M| = 0\}$

Ω is Not a vector space

Counter Ex] $M_1 = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ $\Rightarrow |M_1| = |M_2| = 0$

and $|M_1 + M_2| = \left|\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right| = 4 - 1 = 3 \neq 0$ so $M_1 + M_2 \notin \Omega$

so Ω is not closed under addition.

Sec 3.5 cont

20) $\Omega = \{$The set of all continuous functions f defined on the interval $[0,1]$ such that $f(0)=1\}$

Ω is not a vector space

Counter Ex) $f_1 = 1 \Rightarrow f_1(0) = 1$

$f_2 = 1 + x \quad f_2(0) = 1$

$$f_1 + f_2 = 2 + x \Rightarrow (f_1 + f_2)(0) = 2 + 0 = 2$$

$\Rightarrow f_1 + f_2 \notin \Omega$ so Ω is not closed under addition.

Alternatively, its easy to see that $f = 0$ is not in Ω so it can't be a vector space

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64) $y'' + \sin y = 0$

Let $\Omega = \{ y$ such that $y'' + \sin y = 0 \}$

Counter Example) Let $y_1 = \pi \Rightarrow y_1'' + \sin y_1 = 0 + \sin(\pi)$
$= 0 + 0 = 0$

$\Rightarrow y_1 \in \Omega$

Let's look at $y_2 = \frac{1}{2}\pi = \frac{1}{2} y_1$

$\Rightarrow y_2'' + \sin y_2 = 0 + \sin(\frac{1}{2}\pi) = 1 \neq 0$

so Ω is not closed under scalar multiplication.

Sec 3.6: 2, 9, 14, 24, 50, 64

2) $V = \mathbb{R}^3$; $S = \{ [1,0,0], [0,1,0], [2,3,1] \}$

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad |A| = 1 \neq 0$ so A is invertible and the columns of A span $\mathbb{R}^3$ since they are linearly independent and 3-dimensional. (See page 151)

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9) $V = \mathbb{R}^3$ $\quad S = \{ [1,0,0], [1,1,0], [1,1,1] \}$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$ $|A| = 1 \neq 0$ so A is invertible and its columns are linearly independent.

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14) $V = \mathbb{P}_2$; $S = \{ t, 1-t \}$

$$W = \left| \begin{bmatrix} t & 1-t \\ 1 & -1 \end{bmatrix} \right| = -t - (1-t) = -t - 1 + t$$
$$= -1 \neq 0$$

so t and $1-t$ are linearly independent

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24) $S = \{ e^t, e^{-t}, \cosh t \}$, $\quad \cosh t = \frac{e^t + e^{-t}}{2}$

So $2\cosh t - e^t - e^{-t} = 0$ so the vectors are linearly dependent.

50 $V=\mathbb{R}^3$, $S=\{[1,0,1],[1,1,0],[0,1,1]\}$

Let $A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow |A| = 1+1-0 = 2 \neq 0$

So A is invertible, which means its columns form a basis for $\mathbb{R}^3$.

64

$$3x_1 + 6x_3 + 3x_4 + 9x_5 = 0$$
$$x_1 + 3x_2 - 4x_3 - 8x_4 + 3x_5 = 0$$
$$x_1 - 6x_2 + 14x_3 + 19x_4 + 3x_5 = 0$$

$$\left[\begin{array}{ccccc|c} 3 & 0 & 6 & 3 & 9 & 0 \\ 1 & 3 & -4 & -8 & 3 & 0 \\ 1 & -6 & 14 & 19 & 3 & 0 \end{array}\right] \xrightarrow[\substack{R_1-R_2\to R_2 \\ R_1-R_3\to R_3}]{\frac{1}{3}R_1\to R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 3 & 0 \\ 0 & -3 & 6 & 9 & 0 & 0 \\ 0 & 6 & -12 & -18 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{3}R_2\to R_2]{2R_2+R_3\to R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 3 & 0 \\ 0 & 1 & -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \quad x_1 = -2x_3 - x_4 - 3x_5$$
$$x_2 = 2x_3 + 3x_4$$

So we can think of x_3, x_4, and x_5 as "free variables", as they can be any numbers and x_1 and x_2 depend on them so the solution space is 3-dimensional, and a basis is $S = \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.