**Problem 17 in Section 2.5:** A rumor is spreading on a college campus with $L = 80\,000$ individuals. At $t = 0$, one thousand individuals have heard the rumor. Set $x_0 = 1\,000$. One day later, $x_1 = 10\,000$ individuals have heard the rumor. Assume that the rate with which the rumor spreads is proportional to the product of the number of people who know it, and the number of people who do not yet know it. Let $x(t)$ denote the number of individuals who have heard the rumor at time $t$ (measured in days). Determine $x(t)$.

**Solution:** The assumption is that

$$x'(t) = \underset{\text{unknown constant}}{s} \times \underset{\text{\#people who have heard}}{x(t)} \times \underset{\text{\#people who have not heard}}{(L - x(t))}.$$

We do not know $s$, but we are given that $x(0) = 1\,000$ and $x(1) = 10\,000$. Setting $r = s\,L$, we obtain that standard form of the logistic equation:

$$\frac{dx}{dt} = r\,x\left(1 - \frac{x}{L}\right).$$

First we determine the general solution of the equation. We separate variables:

$$\frac{dx}{x\,(1 - x/L)} = r\,dt. \tag{1}$$

Noting that

$$\frac{1}{x\,(1 - x/L)} = \frac{1}{x} + \frac{1/L}{1 - x/L}$$

we can easily integrate (1)

$$\log(x) - \log(1 - x/L) = r\,t + C.$$

(Note that since $0 < x < L$, no absolute values are needed.) Simplifying:

$$\log\frac{x}{1 - x/L} = r\,t + C \quad \Rightarrow \quad \frac{x}{1 - x/L} = e^{r\,t + C} \quad \Rightarrow \quad \frac{x}{1 - x/L} = B\,e^{r\,t} \tag{2}$$

where we set $B = e^C$. Solve for $x$ we finally get

$$x(t) = \frac{B\,e^{r\,t}}{1 + \frac{1}{L}\,B\,e^{r\,t}} = \frac{L}{1 + (L/B)\,e^{-r\,t}}.$$

Note that $x(t) \to L$ as $t \to \infty$.

Next we determine $B$ and $r$ from the given values of $x$. From the last equality in equation (2) we get immediately that

$$B = \frac{1}{e^{r\,0}}\,\frac{x_0}{1 - x_0/L} = \frac{x_0}{1 - x_0/L}$$

Inserting $t = 1$ in the same equation, and using the value of $B$ we also get

$$\frac{x_1}{1 - x_1/L} = \frac{x_0}{1 - x_0/L}\,e^{r\,1} \qquad \Rightarrow \qquad r = \log\frac{\frac{x_1}{1 - x_1/L}}{\frac{x_0}{1 - x_0/L}} = \log\frac{\frac{10\,000}{1 - 10\,000/80\,000}}{\frac{1\,000}{1 - 1\,000/80\,000}} = 2.4235\cdots$$

Aggregating, we find that

$$x(t) = \frac{L}{1 + (L/(x_0/(1 - x_0/L)))\,e^{-r\,t}} = \frac{L}{1 + ((L/x_0) - 1)\,e^{-r\,t}} \approx \frac{80\,000}{1 + 79\,e^{-2.4235\,t}}.$$