Example Consider the ODE

$$t^3 y^{(3)} - t^2 y^{(2)} + 2t y' - 2y = 0 \qquad (*)$$

on the interval $I = (1,3)$.

Set $y_1(t) = t \quad y_2(t) = t\log(t) \quad y_3 = t^2$

Prove that any sol^n of $(*)$ takes the form

$$y(t) = c_1 y_1 + c_2 y_2 + c_3 y_3$$

Sol^n We need to prove that $S = \{y_1, y_2, y_3\}$ forms a basis for the sol^n space $V = \{y \in C^3(I) : y \text{ solves } (*)\}$.

The existence thm applies since $(*)$ can be written

$$y''' - \frac{1}{t} y'' + \frac{2}{t^2} y' - \frac{2}{t^3} y = 0. \quad \leftarrow \text{All coefficient functions are continuous on } I!$$

(Note that $t^3 \neq 0$ when $t \in I$ so it is ok to divide by t^3.)

We consequently know that $\dim(V) = 3$ so all we need to prove is
(a) y_1, y_2, y_3 all solve $(*)$
(b) $S = \{y_1, y_2, y_3\}$ is linearly independent.

Proof of (a):

$$t^3 y_1''' - t^2 y_1'' + 2t y_1' - 2y_1 = t^3 \cdot 0 - t^2 \cdot 0 + 2t \cdot 1 - 2t = 0 \quad \text{ok!}$$

$$t^3 y_2''' - t^2 y_2'' + 2t y_2' - 2y_2 = t^3\left(-\frac{1}{t^2}\right) - t^2\left(\frac{1}{t}\right) + 2t(1 + \log(t)) - 2t\log(t) = 0 \quad \text{ok!}$$

$$t^3 y_3''' - t^2 y_3'' + 2t y_1' + - 2y_1 = t^3 \cdot 0 - t^2 2 + 2t \cdot 2t - 2t^2 = 0 \quad \text{ok!}$$

Proof of (b): We form the Wronskian:

$$W(t) = \det\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \det\begin{bmatrix} t & t\log(t) & t^2 \\ 1 & 1 + \log(t) & 2t \\ 0 & 1/t & 2 \end{bmatrix} =$$

$$= t(1 + \log(t))2 + 0 + 2t - 2t - 2t\log(t) - 0 = t$$

so $W(t) = 0$ when $t \in I \Rightarrow \{y_1, y_2, y_3\}$ is linearly indep!

Consider the linear n'th order eq^n

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \quad (1)$$

a_i's are constants here, not functions.

The characteristic eq^n is

$$r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

If the char. eq^n has n distinct roots $r_1, r_2, \ldots, r_n$, then the general sol^n of (1) is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$$

<u>Example</u> Construct the general sol^n of $y'''' - 16y = 0$.

<u>Sol^n</u> The characteristic eq^n is $r^4 - 16 = 0$ which has the 4 distinct roots $r_1 = 2$ $r_2 = -2$ $r_3 = 2i$ $r_4 = -2i$.

The general sol^n is $y(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3 e^{2it} + c_4 e^{-2it}$.

A purely real basis is obtained by setting $\left.\begin{aligned} d_3 &= c_3 + c_4 \\ d_4 &= ic_3 - ic_4 \end{aligned}\right\} \Rightarrow$

$$\Rightarrow y(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3 \cos(2t) + c_4 \sin(2t).$$

<u>Example</u> Construct the general sol^n of $y''' + 5y'' + 3y' - 9 = 0$.

Hint: One soln is e^t!

<u>Sol^n</u> The characteristic eq^n is $r^3 + 5r^2 + 3r - 9 = 0. \quad (2)$

~~We have~~ $\cancel{0 = r^3 + 5r^2 + 3r - 9 =}$

The hint tells us that $r_1 = 1$ is a root of (2).

We get $r^3 + 5r^3 + 3r - 9 = (r-1)(r^2 + 6r + 9) = 0$.

The other two roots are then $r_{2,3} = -3 \pm \sqrt{(-3)^2 - 9} = -3$.

Since this is a double root, the general sol^n is

$$y(t) = c_1 e^{-t} + c_2 e^{-3t} + c_3 t e^{-3t}$$