SECTION 4.5 - VARIATION OF PARAMETERS

Suppose that we seek a particular sol^n to the eq^n

$$y'' + y = \frac{1}{\cos(t)} \quad \text{for} \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

The methods described so far do not help.

A particular sol^n is $\quad y_p = \cos(t)\log(\cos(t)) + t\sin(t)$.

How could you possibly find this !?!?

The answer is a method called "variation of parameters".

Basic idea: Find the general sol^n to the homogeneous eq^n and then look for a particular sol^n by "perturbing" it.

To be specific, let p & q be functions and consider the eq^n

$$y'' + py' + qy = f \qquad \text{(ODE)}$$

Suppose y_1 & y_2 are linearly independent sol^{ns} of

$$y_j'' + p y_j' + q y_j = f. \qquad \text{(HOM)}$$

In other words, $\{y_1, y_2\}$ is a basis for the sol^n space of the homogeneous eq^n.

Now, somewhat arbitrarily, we look for a particular sol^n of the form

$$y_p(t) = v_1(t)\,y_1(t) + v_2(t)\,y_2(t) \qquad \text{(ANSATZ)}$$

where $v_1 = v_1(t)$ & $v_2 = v_2(t)$ are two functions to be determined.

By plugging y_p into (ODE), we get one eq^n for y_p, but this is not enough since we've got two unknown functions.
We add an "ad hoc" equation

$$v_1' y_1 + v_2' y_2 = 0. \qquad (1)$$

Time to start calculating...

From (ANSATZ), we get

$$y_p' = \underbrace{v_1' y_1 + v_2' y_2}_{=0 \text{ by } (1)} + v_1 y_1' + v_2 y_2' = v_1 y_1' + v_2 y_2' \qquad (2)$$

$$y_p'' = v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2'' \qquad (3)$$

Insert (2) & (3) into (ODE):

$$\underbrace{v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2''}_{=y_p''} + \underbrace{p v_1 y_1' + p v_2 y_2'}_{=p y_p'} + \underbrace{q v_1 y_1 + q v_2 y_2}_{=q y_p} = f \Rightarrow$$

$$\Rightarrow v_1' y_1' + v_2' y_2' + v_1 \underbrace{(y_1'' + p y_1' + q y_1)}_{=0 \text{ since } y_1 \text{ solves (HOM)}} + v_2 \underbrace{(y_2'' + p y_2' + q y_2)}_{=0 \text{ since } y_2 \text{ solves (HOM)}} = f$$

To summarize: from (ODE), we obtained the cond^n

$$v_1' y_1' + v_2' y_2' = f \qquad (4).$$

We write (1) & (4) as a system for v_1' & v_2':

$$\begin{cases} y_1' v_1' + y_2' v_2' = f \\ y_1 v_1' + y_2 v_2' = 0 \end{cases} \iff \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

Use the formula for the inverse of a 2x2 matrix,

with $W(t) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$ ← Wronskian of $\{y_1, y_2\}$!

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \frac{1}{W(t)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix},$$

So $v_1' = -\dfrac{y_2 f}{W}$ and $v_2' = \dfrac{y_1 f}{W}$

Now simply integrate to find v_1 and v_2!

Let us summarize our procedure for finding a particular sol^n. to $y'' + py' + qy = f$

Step 1 Find two independent sol^{ns} y_1 and y_2 of the homogeneous eq^n $y'' + py' + qy = 0$

Step 2: Compute the Wronskian $W(t) = y_1 y_2' - y_1' y_2$

Step 3: Integrate the equations $\begin{cases} v_1' = -\dfrac{y_2 f}{W} \\ v_2' = \dfrac{y_1 f}{W} \end{cases}$ to obtain v_1 and v_2.

Step 4: Aggregate the sol^n $y_p = v_1 y_1 + v_2 y_2$

Remarks * If you are given an eq^n of the form

$$r(t)\,y''(t) + p(t)\,y'(t) + q(t)\,y(t) = f(t),$$

then start by dividing by $r(t)$!

$$y''(t) + \frac{p(t)}{r(t)}\,y'(t) + \frac{q(t)}{r(t)}\,y(t) = \frac{f(t)}{r(t)}$$

* Note that $W(t) \neq 0$ since $\{y_1, y_2\}$ is a linearly independent set of sol^{ns} to an ODE. (Recall that whereas for a general set of functions it is not the case that linear independence implies non-zero Wronskian, this is the case when all functions solve a joint ODE.)

* When integrating the eq^{ns}

$$v_1' = -\frac{y_2 f}{W} \quad \text{and} \quad v_2' = \frac{y_1 f}{W}$$

there is no need to include general integration constants - any sol^n will do. (This is similar to the 'integrating factor' situation.)

~~DE 117~~

D.E. 4.5-e

Example $y'' + y = \frac{1}{\cos t}$ $\quad |t| < \pi/2$

$r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_1 = \cos(t) \quad y_2 = \sin(t)$

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = \cos^2(t) + \sin^2(t) = 1$$

$$v_1' = -\frac{y_2 f}{W} = -\sin(t)\frac{1}{\cos(t)} = -\tan(t) \Rightarrow v_1 = \log(\cos(t))$$

$$v_2' = \frac{y_1 f}{W} = \frac{\cos(t)\frac{1}{\cos(t)}}{1} = 1 \Rightarrow v_2 = t$$

$$y_p = v_1 y_1 + v_2 y_2 = \cos(t)\log(\cos(t)) + t\sin(t)$$

(13) $t^2y'' - 2ty' + 2y = t^3\sin(t)$ ~~DE 118~~ D.E. 4.5 f

$y_1 = t \quad y_2 = t^2$

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = t\sin(t) \qquad f = t\sin(t)$$

$$W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2$$

$$v_1' = -\frac{y_2 f}{W} = -\frac{t^2\, t\sin(t)}{t^2} = -t\sin(t)$$

$$v_1 = -\int t\sin(t)\,dt = t\cos(t) - \int \cos(t)\,dt = t\cos(t) - \sin(t)$$

$$v_2' = \frac{y_1 f}{W} = \frac{t\, t\sin(t)}{t^2} = \sin(t) \Rightarrow v_2 = -\cos(t)$$

$$y_p = v_1y_1 + v_2y_2 = t^2\cos(t) - t\sin(t) - t^2\cos(t) = -t\sin(t)$$

Check:
$$y' = \sin(t) + t\cos(t)$$
$$y'' = \cos(t) + \cos(t) - t\sin(t) = 2\cos(t) - t\sin(t)$$

$$t^2y'' - 2ty' + 2y = \cancel{2t^3c} - t^3s\ \cancel{-2t^2s} - \cancel{2t^3c} + \cancel{2t^2s}$$

$$v_1'y_1' + v_2'y_2' = -t\sin(t) + \sin(t)\,2t = t\sin(t)$$

$$\boxed{y = c_1t + c_2t^2 - t\sin t}$$

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~~DE 4.4~~

D.E. 4.5g

$$y''+5y'+6y=\cos e^t$$

$r^2+5r+6=0 \qquad y_1=e^{-2t} \qquad y_2=e^{-3t}$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix}=-3e^{-5t}+2e^{-5t}=-e^{-5t}$$

$$V_1'=-\frac{y_2 f}{W}=\frac{e^{-3t}\cos e^t}{e^{-5t}}=e^{2t}\cos e^t$$

$$V_1=\int e^{2t}\cos e^t\,dt=\left\{\begin{matrix} x=e^t \\ dx=e^t dt\end{matrix}\right\}=\int x^2\cos x\,\frac{dx}{x}=\int x\cos x\,dx=$$

$$=x\sin x-\int \sin x\,dx=x\sin x+\cos x=e^t\sin e^t+\cos e^t$$

$$V_2'=\frac{y_1 f}{W}=\frac{e^{-2t}\cos e^t}{-e^{-5t}}=-e^{3t}\cos e^t$$

$$V_2=-\int e^{3t}\cos e^t\,dt=-\int x^2\cos x\,dx=-x^2\sin x+\int 2x\sin x\,dx=$$

$$=-x^2\sin x-2x\cos x+\int 2\cos x\,dx=-x^2\sin x-2x\cos x+2\sin x$$

$$y_p=V_1y_1+V_2y_2=(x\sin x+\cos x)\frac{1}{x^2}+(-x^2\sin x-2x\cos x+2\sin x)\frac{1}{x^3}=$$

$$=\cancel{\frac{1}{x}\sin x}+\frac{1}{x^2}\cos x-\cancel{\frac{1}{x}\sin x}-\frac{2}{x^2}\cos x+\frac{2}{x^3}\sin x=-\frac{1}{x^2}\cos x+\frac{2}{x^3}\sin x$$

$$\boxed{y_p=-e^{-2t}\cos e^t+2e^{-3t}\sin e^t}$$