Thm Let A be an $n \times n$ matrix.
Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be different evals of A, and let $v_1, v_2, \dots, v_k$ be the associated evecs.
Then the set $\{v_1, v_2, \dots, v_k\}$ in linearly indep.

Proof: Set $\Omega_j = \{v_1, v_2, \dots, v_j\}$.
We will prove that each Ω_j is lin. indep by induction.

INDUCTION ASSUMPTION: Ω_j is linearly indep.
The assumption is obviously true for $j=1$!
Now assume that it is true up to $j-1$.
We need to prove that it is true for j.

Suppose $c_1 v_1 + c_2 v_2 + \dots + c_j v_j = 0$. (1)

Multiply (1) by A: $c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_j \lambda_j v_j = 0$ (2)

Multiply (1) by λ_j: $c_1 \lambda_j v_1 + c_2 \lambda_j v_2 + \dots + c_j \lambda_j v_j = 0$ (3)

Subtract (3) from (2): $c_1(\lambda_1 - \lambda_j) v_1 + c_2(\lambda_2 - \lambda_j) v_2 + \dots + c_{j-1}(\lambda_{j-1} - \lambda_j) v_{j-1} = 0$ (4)

Since $\{v_1, v_2, \dots, v_{j-1}\}$ is a linearly indep set, and $\lambda_i \neq \lambda_j$ when $i \neq j$, equation (4) implies that $c_1 = c_2 = \dots = c_{j-1} = 0$.
Then (1) implies that $c_j = 0$ ~~set~~ so (1) holds only if all c_j's are zero!
This proves that $\{v_1, v_2, \dots, v_j\}$ is linearly indep.