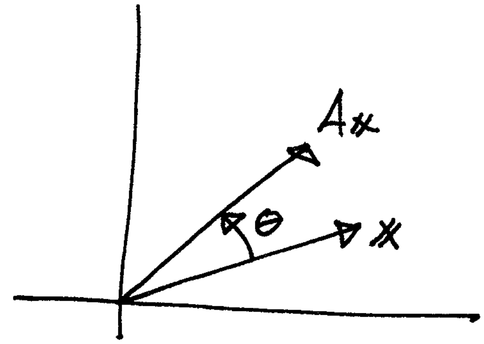
Example $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



Suppose $\sin\theta \neq 0$!

~~How~~ Note that $\mathbf{x}$ and $A\mathbf{x}$ cannot possibly be parallel!
How could A have any evals or evecs?
Let us proceed as usual.

Step 1 $p_A(\lambda) = \det\begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta$

Step 2 $p_A(\lambda) = 0 \Rightarrow (\cos\theta - \lambda)^2 = -\sin^2\theta \Rightarrow \lambda = \underbrace{\cos\theta \pm i\sin\theta}_{= e^{\pm i\theta}}$

The evals are complex numbers!

Say $\lambda_1 = e^{i\theta}$ $\quad \lambda_2 = e^{-i\theta}$

Step 3 * Process $\lambda_1 = e^{i\theta}$

$$\left[\begin{array}{cc|c} \cos\theta - e^{i\theta} & -\sin\theta & 0 \\ \sin\theta & \cos\theta - e^{-i\theta} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -i\sin\theta & -\sin\theta & 0 \\ \sin\theta & -i\sin\theta & 0 \end{array}\right] \overset{\sin\theta \neq 0}{\sim} \left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

We get $v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

Verify! $Av_1 = \begin{bmatrix} i\cos\theta - \sin\theta \\ i\sin\theta + \cos\theta \end{bmatrix} = \begin{bmatrix} i(\cos\theta + i\sin\theta) \\ \cos\theta + i\sin\theta \end{bmatrix} = e^{i\theta}\begin{bmatrix} i \\ 1 \end{bmatrix} = \lambda_1 v_1$ Ok!

* Process $\lambda_2 = e^{-i\theta}$

$$\left[\begin{array}{cc|c} \cos\theta - e^{-i\theta} & -\sin\theta & 0 \\ \sin\theta & \cos\theta - e^{-i\theta} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} i\sin\theta & -\sin\theta & 0 \\ \sin\theta & i\sin\theta & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$$\boxed{\text{Answer: } \lambda_1 = e^{i\theta} \quad v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad \lambda_2 = e^{-i\theta} \quad v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}}$$