Example Consider the ODE

$$\begin{cases} x' = 33 - 10x - 3y + x^2 \\ y' = -18 + 6x + 2y - xy \end{cases}$$

The eq^n has an equilibrium at $(4,3) = (x_0, y_0)$.
Determine its type!
(No need to draw the phase diagram.)

Sol^n

$$f = 33 - 10x - 3y + x^2$$
$$g = -18 + 6x + 2y - xy$$

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} -10+2x & -3 \\ 6-y & 2-x \end{bmatrix}$$

Insert $x = 4$ & $y = 3$: $J = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}$

Find evals of J!

$$0 = p_J(\lambda) = \det\begin{bmatrix} -2-\lambda & -3 \\ 3 & -2-\lambda \end{bmatrix} = (-2-\lambda)(-2-\lambda) + 9 = \lambda^2 + 4\lambda + 13$$

Evals are $\lambda_{1,2} = -2 \pm \sqrt{(-2)^2 - 13} = -2 \pm \sqrt{-9} = -2 \pm 3i$

Answer $(4,3)$ is a stable spiral point!

Example $\dot{x} = y - x^3$

$\dot{y} = -x - y^3$

What does linear analysis say about the equilibrium at $(0,0)$?

Sol^n

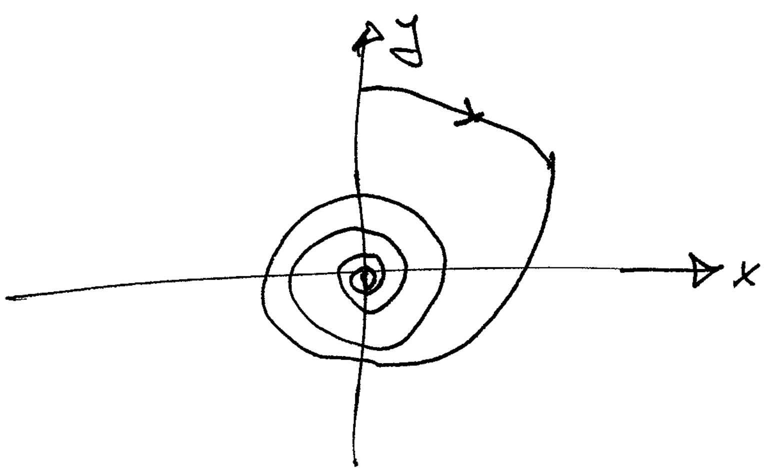
$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$P_J(\lambda) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1$$

so $\lambda = \pm i$

Linear analysis cannot tell!

(The answer is that it is an asymptotically stable spiral point)



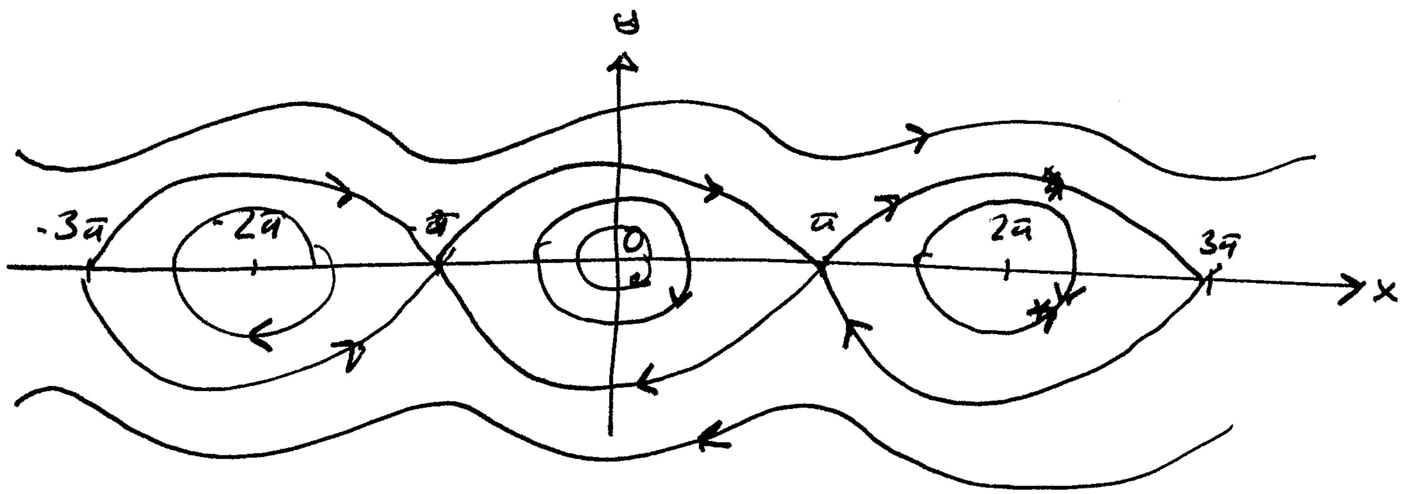
<u>Example</u> first review the phase diagram of

$$\Theta'' + \sin\Theta = 0$$

$$x = \Theta \quad \Rightarrow \quad x' = \Theta' = y$$

$$y = \Theta' \qquad y' = \Theta'' = -\sin\Theta = -\sin x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -\sin x \end{bmatrix}$$



Now consider $\Theta'' + \Theta' + \sin\Theta = 0$

Damping!

$$x = \Theta \qquad \begin{cases} x' = \Theta' = y = f \\ y' = \Theta'' = -\Theta' - \sin\Theta = -y - \sin x = g \end{cases}$$

$$y = \Theta'$$

$f = 0 \Rightarrow y = 0$

$g = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi$

$$J = \begin{bmatrix} 0 & 1 \\ -\cos x & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(-1)^n & -1 \end{bmatrix}$$

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If n is even: $J = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$

$$0 = -\lambda(-1-\lambda)+1 = \lambda^2+\lambda+1 \Rightarrow \lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4}-1} = \frac{-1 \pm i\sqrt{3}}{2}$$

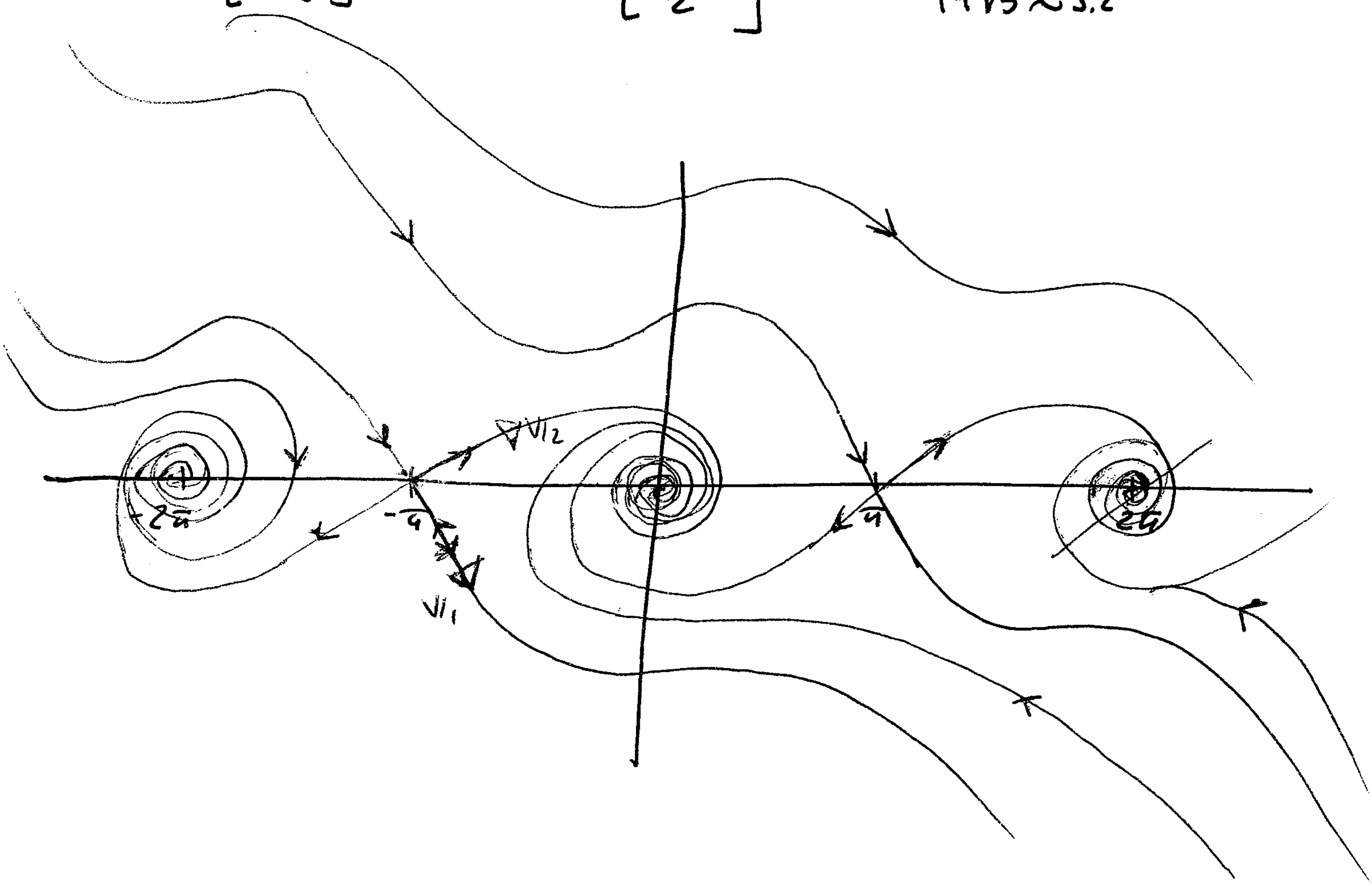
~~Sp~~ Stable spiral point

If n is odd: $J = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

$$0 = -\lambda(-1-\lambda)-1 = \lambda^2+\lambda-1 \Rightarrow \lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+1} = \frac{-1 \pm \sqrt{5}}{2}$$

$\lambda_1 = \frac{-1-\sqrt{5}}{2}$ $\qquad$ $\lambda_2 = \frac{-1+\sqrt{5}}{2}$

$V_1 = \begin{bmatrix} -2 \\ 1+\sqrt{5} \end{bmatrix}$ $\qquad$ $V_2 = \begin{bmatrix} 1+\sqrt{5} \\ 2 \end{bmatrix}$ $\qquad$ $1+\sqrt{5} \approx 3.2$



Sometimes we can construct exact formulas for the trajectories without actually solving the ODE!

Note that if $\frac{dx}{dt} = f(x,y)$ and $\frac{dy}{dt} = g(x,y)$,

then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(x,y)}{f(x,y)}$ ← ODE we can solve for $y = y(x)$!

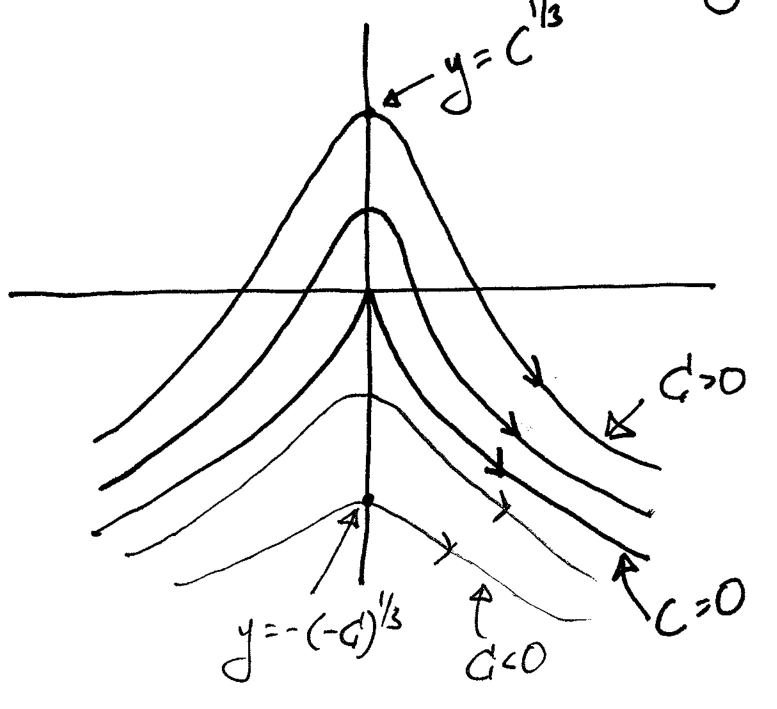
<u>Example</u> $\begin{cases} x' = y^2 \\ y' = -x \end{cases}$ Determine a formula for the trajectories!

<u>Sol^n</u> $\frac{dy}{dx} = -\frac{x}{y^2} \Rightarrow y^2 dy = -x\,dx$

$$\Rightarrow \frac{1}{3} y^3 = -\frac{1}{2} x^2 + C$$

We get one trajectory for each value of C!

For instance: $C = 0 \Rightarrow y^3 = -\frac{1}{2}x^2 \Rightarrow y = -\frac{1}{2}|x|^{2/3}$



Would linearization around $(x_0, y_0) = (0,0)$ work?

$$J = \begin{bmatrix} 0 & 2y \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

J is singular! ($\det(J) = 0$)

Linear analysis is useless!

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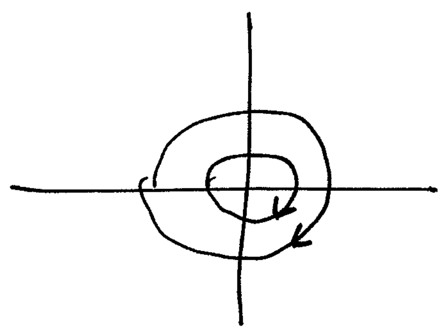
Example $\begin{cases} x' = y \\ y' = -x \end{cases}$ $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$p(\lambda) = \lambda^2 + 1 \Rightarrow \lambda = \pm i$

Linear analysis does not work!

$\frac{dy}{dx} = \frac{-x}{y}$ $\quad y\,dy = -x\,dx \Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$

$\Rightarrow x^2 + y^2 = 2C \Rightarrow$ circles!



Example $\begin{cases} x' = y \\ y' = x \end{cases}$ $\quad \frac{dy}{dx} = \frac{x}{y} \Rightarrow y\,dy = x\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C \Rightarrow y^2 - x^2 = 2C$

hyperbolas!