APPM 2360: Section exam 3
7.00pm – 8.30pm, April 15, 2009.

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) recitation section (4) your instructor's name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

Problem 1: (36 points) Give a brief answer to each question. Box your answer. Each correct answer earns 6 points. No work given for this question will be graded.

(a) A system is described by the equation $x'' + 3\,x' + 5\,x = \sin(2\,t)$. Find the frequency of the steady-state solution (*i.e.* the frequency of the solution as $t \to \infty$).

(b) Let A be a 2×2 matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. Find the eigenvalues of A^5. *Hint:* The eigenvectors of A and A^5 are the same.

(c) A mass of 1 kg is attached to a spring with constant $k = 4$ N/m. There is no damping. The system is forced with a forcing term of the form $f(t) = 0.01\cos(\omega_{\rm f}\,t)$ (measured in Newtons). Initially the mass is at rest at its equilibrium position. Find a value of ω_f that guarantees that the amplitude of the resulting oscillations grows without limit.

(d) If $\mathbf{v_1} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} b \\ d \end{bmatrix}$ are non-zero vectors such that $A\,\mathbf{v}_1 = 2\,\mathbf{v}_1$ and $A\,\mathbf{v}_2 = -\mathbf{v}_2$ for some 2×2 matrix A, then what are the possible values of the rank of the matrix $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

(e) Let A be a matrix, and let $\mathbf{u}_1$ and $\mathbf{u}_2$ be non-zero vectors such that $A\,\mathbf{u}_1 = 2\,\mathbf{u}_1$ and $A\,\mathbf{u}_2 = 2\,\mathbf{u}_2$. Which of the following statements are *necessarily* true:
(1) $\mathbf{u}_1 + \mathbf{u}_2$ is an eigenvector of A with associated eigenvalue 4.
(2) $3\,\mathbf{u}_1$ is an eigenvector of A with associated eigenvalue 2.
(3) $\det(A - 2\,I) = 0$.

(f) Let A be a 3×3 matrix A with real entries. Which of the following statements could *possibly* be true:
(1) $\lambda_1 = 1 + i$ and $\lambda_2 = 3 - i$ are both eigenvalues of A.
(2) A has only one eigenvalue, but three linearly independent eigenvectors.
(3) All eigenvalues of A have non-zero imaginary parts.

PLEASE TURN OVER

For question 2 — 5, motivate your answers. A correct answer with no work may receive no credit, while an incorrect answer with some correct work may result in partial credit.

Problem 2: (18 points) Consider the equation

$$y'' + 2\,y' + 5\,y = 0. \tag{1}$$

(a) (6 points) Construct the general solution of (1).

(b) (6 points) Set $x_1 = y$ and $x_2 = y'$. Specify a matrix A such that $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(c) (6 points) Now consider the equation $z'' + 2\,z' + b\,z = 0$. For which real values of b do there exist solutions such that $z(t) \to \infty$ as $t \to \infty$?

Extra credit: (2 points) What are the eigenvalues of the matrix A that you derived in 2(b)?

Problem 3: (12 points) Consider the equation

$$y'' + 2\,y' - 3\,y = 2\,e^{2\,t}. \tag{2}$$

(a) (6 points) Construct the general solution of (2).

(b) (6 points) Construct the specific solution of (2) that satisfies $y(0) = 0$ and $y'(0) = 1$.

Problem 4: (16 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(a) (6 points) Show that 2 is one eigenvalue of A. Find the other eigenvalues of A.

(b) (10 points) For all eigenvalues of A, find the corresponding eigenvectors and the dimensions of the eigenspaces.

Problem 5: (18 points) Consider the equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3}$$

for the functions $x_1 = x_1(t)$ and $x_2 = x_2(t)$.

(a) (6 points) Find the general solution of (3).

(b) (6 points) Find the particular solution of (3) that satisfies $x_1(0) = 1$ and $x_2(0) = 3$.

(c) (6 points) Let x_1 and x_2 be a solution of (3) such that $x_1(3) = a$ and $x_2(3) = b$. Specify a relationship that a and b must satisfy for it to be the case that $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$.

Hint 1: The matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has the eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
Hint 2: There exists a solution to 5(c) that requires essentially no calculations.