APPM 2360: Midterm 3

7.00pm – 8.30pm, April 16, 2008.

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. Show ALL of your work in your bluebook and box in your final answer. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

Problem 1: (15 points, 3 each) State whether the following statements are (always) "TRUE" or "FALSE" (meaning not always true). You must write the full word TRUE or FALSE. For this question, you do not need to show your work or reasoning.

(a) A 3×3 real matrix A can have three complex eigenvalues.

(b) All identity matrices have the same eigenvalues independently of their dimension.

(c) If m, b, and k are real positive numbers, then any solution of the equation $m\,\ddot{x}+b\,\dot{x}+k\,x=0$ tends to zero as $t \to \infty$.

(d) An $n \times n$ real matrix A and its transpose A^{T} have the same eigenvectors.

(e) If $W(y_1, y_2)$ is the Wronskian of two functions, then $W(yy_1, yy_2) = y^2 W(y_1, y_2)$.

Problem 2: (10 points, 2 each) Consider the equation $\ddot{y} + \dot{y} - 6\,y = f$. For each of the following choices of f, give the form of a particular solution to the equation. You do not need to motivate your answers, or determine any constants. (Sample answer: "$y_{\mathrm{p}} = \alpha\,t + \beta\,\cos(t)$ where α and β are constants".)

(a) $f(t) = e^{2\,t}$

(b) $f(t) = e^{-2\,t}$

(c) $f(t) = t\,e^{t}$

(d) $f(t) = \cos(2\,t) + t^2$

(e) $f(t) = t\,\sin(5\,t)$

Problem 3: (25 points) Consider the equation $\ddot{x} + 2\,\dot{x} + 5\,x = 0$.

(a) (6 points) Construct the general solution.

(b) (6 points) Construct the solution that satisfies the initial conditions $x(0) = 1$ and $\dot{x}(0) = 3$.

(c) (7 points) Let ω be a real number and consider the equation

$$\ddot{x} + 2\,\dot{x} + 5\,x = \cos(\omega\,t).$$

Prove that a particular solution is given by $x_{\mathrm{p}} = C(\omega)\,\cos(\omega\,t - \delta)$ where $C(\omega) = 1/\sqrt{(5-\omega^2)^2 + 4\,\omega^2}$ and $\tan\delta = 2\,\omega/(5-\omega^2)$.

(d) (6 points) Find the value (or values) of ω for which the function $C(\omega)$ in part (c) attains its maximum. What happens to $C(\omega)$ as $\omega \to \infty$?

You may find some of the following identities useful:

$$\begin{aligned}
\sin(\alpha-\beta) &= \sin(\alpha)\,\cos(\beta) - \cos(\alpha)\,\sin(\beta)\\
\cos(\alpha-\beta) &= \cos(\alpha)\,\cos(\beta) + \sin(\alpha)\,\sin(\beta)\\
\cos(\alpha)^2 &= 1/(1+\tan(\alpha)^2)
\end{aligned}$$

Problem 4: (20 points) Consider the differential equation

$$\mathbf{x}' = A\,\mathbf{x}, \quad \text{where} \quad A = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} y \\ z \end{bmatrix}.$$

(a) (7 points) Construct the eigenvalues and eigenvectors of A.

(b) (7 points) Construct the general solution $\mathbf{x}$.

(c) (6 points) The function $y = y(t)$ satisfies an equation of the form

$$y'' + a\,y' + b\,y = 0.$$

Determine the numbers a and b.

Problem 5: (15 points) Compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

For 5 points extra credit: Construct all eigenvalues and eigenvectors of A^{-1}.

Problem 6: (15 points) Let n be a non-negative integer and consider for positive t the differential equation

$$t\,y'' - (t+n)\,y' + n\,y = 0.$$

(a) (5 points) Verify that $y_1 = e^t$ is a solution of the equation. (5p)

(b) (10 points) Find a second solution to the equation (linearly independent of the first). You may leave your answer in integral form.
Hint: You may want to look for a solution of the form $y(t) = z(t)\,e^t$.