APPM 2360: Section exam 3

7.00pm – 8.30pm, April 15, 2009.

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) recitation section (4) your instructor's name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

Problem 1: (36 points) Give a brief answer to each question. Box your answer. Each correct answer earns 6 points. No work given for this question will be graded.

(a) A system is described by the equation $x'' + 3\,x' + 5\,x = \sin(2\,t)$. Find the frequency of the steady-state solution (*i.e.* the frequency of the solution as $t \to \infty$).

(b) Let A be a 2×2 matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. Find the eigenvalues of A^5. *Hint:* The eigenvectors of A and A^5 are the same.

(c) A mass of 1 kg is attached to a spring with constant $k = 4$ N/m. There is no damping. The system is forced with a forcing term of the form $f(t) = 0.01\cos(\omega_{\mathrm{f}}\,t)$ (measured in Newtons). Initially the mass is at rest at its equilibrium position. Find a value of ω_{f} that guarantees that the amplitude of the resulting oscillations grows without limit.

(d) If $\mathbf{v_1} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} b \\ d \end{bmatrix}$ are non-zero vectors such that $A\,\mathbf{v}_1 = 2\,\mathbf{v}_1$ and $A\,\mathbf{v}_2 = -\mathbf{v}_2$ for some 2×2 matrix A, then what are the possible values of the rank of the matrix $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

(e) Let A be a matrix, and let $\mathbf{u}_1$ and $\mathbf{u}_2$ be non-zero vectors such that $A\,\mathbf{u}_1 = 2\,\mathbf{u}_1$ and $A\,\mathbf{u}_2 = 2\,\mathbf{u}_2$. Which of the following statements are *necessarily* true:
(1) $\mathbf{u}_1 + \mathbf{u}_2$ is an eigenvector of A with associated eigenvalue 4.
(2) $3\,\mathbf{u}_1$ is an eigenvector of A with associated eigenvalue 2.
(3) $\det(A - 2\,I) = 0$.

(f) Let A be a 3×3 matrix A with real entries. Which of the following statements could *possibly* be true:
(1) $\lambda_1 = 1 + i$ and $\lambda_2 = 3 - i$ are both eigenvalues of A.
(2) A has only one eigenvalue, but three linearly independent eigenvectors.
(3) All eigenvalues of A have non-zero imaginary parts.

Solutions to Problem 1:

(a) Either of the answers "$1/\pi$" and "2" give full credit.

(b) $\lambda_1 = 1^5 = 1$ and $\lambda_2 = 2^5 = 32$

(c) $\omega_{\rm f} = \sqrt{k/m} = 2$

(d) 2

(e) (2) and (3)

(f) (2)

Comments:

(a) The steady state solution takes the form $x(t) = C\,\cos(2\,t - \delta)$ for some numbers C and δ. The "angular velocity" is then $\omega = 2$ and the "frequency" is $f = \omega/(2\,\pi) = 1/\pi$.

(b) You can easily verify that if $A\,\mathbf{v} = \lambda\,\mathbf{v}$, then $A^p = \lambda^p\,\mathbf{v}$.

(c) The governing equation is $m\,\ddot{x} + k\,x = 0.01\,\cos(\omega_{\rm f}\,t)$ which we rewrite as $\ddot{x} + \omega^2\,x = 0.01\,\cos(\omega_{\rm f}\,t)$ with $\omega = \sqrt{k/m} = 2$. Resonance occurs if $\omega_{\rm f} = \omega = 2$.

(d) Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors associated with different eigenvalues, they must be linearly independent. It follows that C is non-singular and has rank 2.

(e) $\mathbf{u}_1 + \mathbf{u}_2$ is indeed an eigenvector, but it has eigenvalue 2 so (1) is not true.

(f) If λ is an eigenvalue of a matrix with real entries, then $\bar{\lambda}$ must also be an eigenvalue. So (1) must be false since otherwise A would have had to have four different eigenvalues ($1+i$, $1-i$, $3-i$, $3+i$) which is impossible. Similarly, (3) must be false since the number of complex eigenvalues must be even (since they come in pairs). To see that (2) can be true, consider the identity matrix.

Problem 2: (18 points) Consider the equation

$$y'' + 2\,y' + 5\,y = 0. \tag{1}$$

(a) (6 points) Construct the general solution of (1).

(b) (6 points) Set $x_1 = y$ and $x_2 = y'$. Specify a matrix A such that $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(c) (6 points) Now consider the equation $z'' + 2\,z' + b\,z = 0$. For which real values of b do there exist solutions such that $z(t) \to \infty$ as $t \to \infty$?

Extra credit: (2 points) What are the eigenvalues of the matrix A that you derived in 2(b)?

Solution:

(a) The characteristic polynomial is $r^2 + 2\,r + 5 = 0$ which has roots $r = -1 \pm 2\,i$.

The general solution is $\boxed{x(t) = b_1\,e^{(-1+2\,i)\,t} + b_2\,e^{(-1-2\,i)\,t}.}$

An alternative (better!) solution is $\boxed{x(t) = c_1\,e^{-t}\,\cos(2\,t) + c_2\,e^{-t}\,\sin(2\,t).}$

(b) We have $x_1' = y' = x_2$ and $x_2' = y'' = -5\,y - 2\,y' = -5\,x_1 - 2\,x_2$. Thus

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ -5\,x_1 - 2\,x_2 \end{bmatrix} = A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The answer is $\boxed{A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}.}$

(c) The characteristic equation is $r^2 + 2\,r + b = 0$ which has roots

$$r_1 = -1 - \sqrt{1-b}, \qquad r_2 = -1 + \sqrt{1-b}.$$

There exists a growing solution precisely if the real part of either r_1 or r_2 is positive. r_1 always has negative real part. r_2 has a positive real part if and only if $b < 0$.

$\boxed{\text{There exists a solution that grows if and only if } b < 0.}$

The above is all the motivation that is required, but we note that the different regimes are:

- If $b > 1$, then r_1 and r_2 are complex numbers with real part -1 so all solutions decay (while oscillating).
- If $b = 1$ then the general solution is $x(t) = c_1\,e^{-t} + c_2\,t\,e^{-t}$. Again, all solutions decay.
- If $0 < b < 1$, then $x(t) = c_1\,e^{r_1\,t} + c_2\,e^{r_2\,t}$ where r_1 and r_2 are real negative numbers. Again, all solutions decay.
- If $b = 0$, then $r_1 < 0$, $r_2 = 0$, and $x(t) = c_1\,e^{r_1\,t} + c_2$. No solution grows to infinity.
- If $b < 0$, then $r_1 < 0$ and $r_2 > 0$ and so $x = c_1\,e^{r_1\,t} + c_2\,e^{r_2\,t}$ grows to infinity if $c_2 > 0$.

Extra credit problem: The eigenvalues are $\lambda_{1,2} = -1 \pm 2\,i$.

Problem 3: (12 points) Consider the equation

$$y'' + 2\,y' - 3\,y = 2\,e^{2\,t}. \tag{2}$$

(a) (6 points) Construct the general solution of (2).

(b) (6 points) Construct the specific solution of (2) that satisfies $y(0) = 0$ and $y'(0) = 1$.

Solution:

(a) The characteristic equation is $r^2 + 2\,r - 3 = 0$ with roots $r_1 = 1$ and $r_2 = -3$. The homogeneous solution is then $y_{\mathrm{h}} = c_1\,e^t + c_2\,e^{-3\,t}$.

Next look for a particular solution of the form $y_{\mathrm{p}} = A\,e^{2\,t}$. We find $y_{\mathrm{p}}'' + 2\,y_{\mathrm{p}}' - 3\,y_{\mathrm{p}} = 5\,A\,e^{2\,t}$ so we must have $A = 2/5$ and so $y_{\mathrm{p}} = (2/5)\,e^{2\,t}$.

$$\boxed{y = y_{\mathrm{h}} + y_{\mathrm{p}} = c_1\,e^t + c_2\,e^{-3\,t} + \frac{2}{5}\,e^{2\,t}.}$$

(b) We construct an equation for c_1 and c_2 from the initial conditions:

$$\begin{aligned} 0 &= y(0) = c_1 + c_2 + 2/5 \\ 1 &= y'(0) = c_1 - 3\,c_2 + 4/5. \end{aligned}$$

Subtract the first equation from the second to obtain

$$1 = -4\,c_2 + 2/5$$

which implies that $c_2 = -3/20$. Then the first equation implies that

$$c_1 = -c_2 - \frac{2}{5} = \frac{3}{20} - \frac{8}{20} = -\frac{5}{20} = -\frac{1}{4}.$$

$$\boxed{y(t) = -\frac{1}{4}\,e^t - \frac{3}{20}\,e^{-3\,t} + \frac{2}{5}\,e^{2\,t}.}$$

Problem 4: (16 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(a) (6 points) Show that 2 is one eigenvalue of A. Find the other eigenvalues of A.

(b) (10 points) For all eigenvalues of A, find the corresponding eigenvectors and the dimensions of the eigenspaces.

Solution:

(a) First we find the characteristic polynomial of A.

$$p_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ -1 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} = (2-\lambda)\,(1-\lambda)^2 - (2-\lambda) = (2-\lambda)\,(\lambda^2 - 2\,\lambda) = -(2-\lambda)^2\,\lambda.$$

The roots of $p(\lambda) = 0$ are clearly $\lambda_1 = 0$ and $\lambda_2 = 2$.

(b) Find the solutions to $(A - 0\,I)\,\mathbf{v} = \mathbf{0}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The solution space is one-dimensional and is spanned by $\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$.

Find the solutions to $(A - 2\,I)\,\mathbf{v} = \mathbf{0}$:

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The solution space is one-dimensional and is spanned by $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Problem 5: (18 points) Consider the equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3}$$

for the functions $x_1 = x_1(t)$ and $x_2 = x_2(t)$.

(a) (6 points) Find the general solution of (3).

(b) (6 points) Find the particular solution of (3) that satisfies $x_1(0) = 1$ and $x_2(0) = 3$.

(c) (6 points) Let x_1 and x_2 be a solution of (3) such that $x_1(3) = a$ and $x_2(3) = b$. Specify a relationship that a and b must satisfy for it to be the case that $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$.

Hint: The matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has the eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution: A simple calculation shows that $A\,\mathbf{v}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\,\mathbf{v}_1$ and $A\,\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1)\,\mathbf{v}_2$ so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$.

(a) The general solution is $\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\, e^{\lambda_1 t}\, \mathbf{v}_1 + c_2\, e^{\lambda_2 t}\, \mathbf{v}_2 = c_1\, e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\, e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.}$

(b) The initial conditions imply that

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix}.$$

Subtracting the second equation from the first we get $-2 = 2\,c_2$ and so $c_2 = -1$. Then the first equation yields $c_1 = 1 - c_2 = 2$.

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2\,e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\,e^{3t} - e^{-t} \\ 2\,e^{3t} + e^{-t} \end{bmatrix}.}$$

(c) Since $\mathbf{x}(t) = c_1\, e^{3t}\, \mathbf{v}_1 + c_2\, e^{-t}\, \mathbf{v}_2$ we see that $x(t) \to 0$ as $t \to 0$ if and only if $c_1 = 0$. This means that the initial conditions must be such that

$$\begin{bmatrix} a \\ b \end{bmatrix} = c_2\, e^{-3} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

In other words, we must have $a = c_2\, e^{-3} = -b$ so the answer is:

$\boxed{x_1(t) \text{ and } x_2(t) \text{ tend to zero as } t \to \infty \text{ if and only if } a = -b.}$

For an alternative solution, simply note that the only characteristics that move in to the origin are those on the line spanned by $\mathbf{v}_2$, which means that

$$\begin{bmatrix} a \\ b \end{bmatrix} = c\,\mathbf{v}_2 = c \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

which again is to say that $a = -b$.