Applied Analysis (APPM 5440): Final Exam
1.30pm – 5.00pm, Dec 11, 2005. Closed books.

In proofs, please state clearly what you assume, and what you will prove.

Problem 1: No motivation is required for the following questions: (2p each)

(a) Define what it means for a subset of a metric space to be *totally bounded*.

(b) Set $I = [0, 1)$. Specify which (if any) of the following inclusions are equalities: $C_{\rm c}(I) \subseteq C_0(I) \subseteq C_{\rm b}(I) \subseteq C(I)$.

(c) Let X be a Hilbert space, and define for $y \in X$ the functional φ_y by setting $\varphi_y(x) = (y, x)$. What do you know about the map $T : X \to X^* : y \mapsto \varphi_y$?

(d) Let $\mathcal{P}$ denote the set of all functions that can be written in the form $f(x) = \sum_{n=0}^{N} \big(a_n \cos(nx) + b_n \sin(nx)\big)$, for some finite integer N, and some complex numbers a_n and b_n. Is $\mathcal{P}$ dense in $C(\mathbb{T})$?

(e) Let $\mathcal{P}$ be as in (d). Is $\mathcal{P}$ dense in $L^2(\mathbb{T})$?

(f) Suppose that $f \in H^k(\mathbb{T})$. Specify for which k, if any, it is necessarily the case that f is continuous.

(g) Consider the metric space X consisting of all rational numbers, equipped with the metric $d(x, y) = |x - y|$. Which of the following sets are open: $A = \{q \in X : 0 < q^2 \leq 4\}$, $B = \{q \in X : 0 < q^2 \leq 2\}$, $C = \{q \in X : 0 < q < \infty\}$.

(h) Let X be a normed linear space, and let X^* define the (topological) dual of X. Define what it means for a sequence $(y_n)_{n=1}^{\infty} \subseteq X^*$ to converge in the weak-$*$ topology.

Problem 2: Let X be a finite-dimensional linear space, and let $||\cdot||_1$ and $||\cdot||_2$ be two norms on X.

(a) Prove that there exist numbers c and C such that $0 < c \leq C < \infty$, and

(1) $$c||x||_2 \leq ||x||_1 \leq C||x||_2, \qquad \forall x \in X.$$

(3p)

(b) Let G be a subset of X. Define what it means for G to be open in the topology generated by the norm $||\cdot||_1$. (2p)

(c) Prove that if G is open in the topology generated by the norm $||\cdot||_1$, then G open in the topology generated by the norm $||\cdot||_2$. (You may use the inequality (1) regardless of whether you answered part (a).) (2p)

Problem 3: Set $I = [-1, 1]$, and consider the functions $f, g_1, g_2 \in C(I)$, given by $f(x) = x^2$, $g_1(x) = 1$, and $g_2(x) = x$. Set $A = \text{span}(g_1, g_2)$. Determine $\alpha = \text{dist}(A, f) = \inf_{g \in A} ||g - f||$. Is the minimizer unique? (4p)

Problem 4: Set $I = [0, 1]$, let k be a continuous function on I^2, and consider the integral operator $T : C(I) \to C(I)$, given by

$$[Tf](x) = \int_0^1 k(x, y)\, f(y)\, dy.$$

Prove that T is compact. (4p)

Problem 5: Let $X = l^1(\mathbb{N})$, and let $(\alpha_n)_{n=1}^{\infty}$ be numbers such that $|\alpha_n| \leq 2^{-n}$. Define the linear operator $T : X \to X$ by setting, for $x = (x_1, x_2, \dots)$, $(Tx)_j = \alpha_j x_1 + x_j$.

(a) Determine $\sup\{\frac{||Tx||}{||x||} : x \neq 0\}$. (3p)

(b) What is the range of T? (1p)

(c) Determine $\sup\{\frac{||x||}{||Tx||} : x \neq 0\}$. (2p)

Problem 6: Let f be a bounded continuous function on $\mathbb{R}^2$ for which there exists a finite number C such that

$$|f(t, a) - f(t, b)| \leq C|a - b|, \qquad \forall\ t, a, b \in \mathbb{R}.$$

Consider the ODE

$$\text{(ODE)} \qquad \begin{cases} \dot{u}(t) = f(t, u(t)), \\ u(0) = 1. \end{cases}$$

State the contraction mapping theorem, and use it to prove that for some $\varepsilon > 0$, the equation (ODE) has a unique solution in $C^1([-\varepsilon, \varepsilon])$. (You do not need to give an optimal ε.) (5p)

Problem 7: Let X be a separable infinite-dimensional Hilbert space. Prove that there exists a family of closed linear subspaces $\{\Omega_t : t \in [0, 1]\}$ such that Ω_s is a strict subset of Ω_t whenever $s < t$. (4p)