Homework set 3 — APPM5440

From the textbook: 1.17, 1.18, 1.20, 1.21, 1.22.

Note that you can find solutions of many textbook problems at:

`http://www.math.ucdavis.edu/~bxn/`

Problem: Suppose that $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are Cauchy sequences in a metric space. Prove that $(d(x_n, y_n))_{n=1}^{\infty}$ converges.

Problem: In the proof that every metric space has a completion, we did not prove that the function $\tilde{d}$ is indeed a metric. Verify that this is the case.

Optional problem: In the proof that every metric space has a completion, the technique we used to prove that $\tilde{X}$ is complete is called the "Cantor diagonal argument". This is a standard technique. Try to use it to prove that the real numbers are not countable. Hint: Assume that there exists an enumeration $(r^{(n)})_{n=1}^{\infty}$ of all real numbers in the interval $(0, 1)$. Suppose that each $r^{(n)}$ has a binary number expansion

$$r^{(n)} = 0.b_1^{(n)}\, b_2^{(n)}\, b_3^{(n)} \dots$$

(so that each $b_j^{(n)}$ is either 0 or 1) and use the "diagonal" technique to construct a real number that is not in the sequence. (There is a full solution in the Wikipedia article on Cantor's diagonal argument.)