Applied Analysis (APPM 5440): Final Exam

7.30am – 10.00am, Dec. 20, 2006. Closed books.

Problem 1: The following problems are worth 2p each. No motivation required for (a), (e), or (f). For the rest, please motivate your answer briefly (at most one or two sentences).

(a) State the contraction mapping theorem.

(b) Let H be a Hilbert space, and let Ω be a subset of H. Under what conditions is it necessarily the case that $H = \Omega \oplus \Omega^{\perp}$?

(c) Given a separable Hilbert space H, and an orthonormal sequence $(e_n)_{n=1}^{\infty}$, specify two different conditions that guarantee that $(e_n)_{n=1}^{\infty}$ is a basis for H.

(d) Let H be a Hilbert space, and let $\varphi \in H^*$. Set $N = \ker(\varphi)$. Suppose that $x, y \in N^{\perp}$. Is it necessarily the case that $|(x,\, y)| = ||x||\,||y||$?

(e) State the Arzelà-Ascoli theorem.

(f) Let (X, d) be a complete metric space and let Ω be a subset of X. Is it true that Ω is pre-compact if and only if it is totally bounded?

(g) Let X be a set with a finite number of elements. Is it true that all metrics on X induce the same topology?

(h) Is the function $f(x) = \sqrt{x}\, \sin \frac{1}{x}$ uniformly continuous on the interval $I = (0,\, 1]$?

Problem 2: Set $I = [0,\, 1]$ and consider the Banach space $X = C_{\mathrm{b}}(I)$ (with the usual norm, $||f||_{\mathrm{u}} = \sup_{x\in I} |u(x)|$). Let $Y = \mathbb{R}$ with the standard norm.

(a) Let $A_n \in \mathcal{B}(X, Y)$. Define what it means for the sequence $(A_n)_{n=1}^{\infty}$ to converge strongly in $\mathcal{B}(X, Y)$. (1p)

(b) Let $A_n \in \mathcal{B}(X, Y)$. Define what it means for the sequence $(A_n)_{n=1}^{\infty}$ to converge in norm in $\mathcal{B}(X, Y)$. (1p)

(c) Let $(A_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{B}(X, Y)$ such that for every $\varepsilon > 0$, there exists an integer N such that $||A_n - A_m|| < \varepsilon$ when $m, n \geq N$. Is it necessarily the case that there exists a unique $A \in \mathcal{B}(X, Y)$ such that the sequence $(A_n)_{n=1}^{\infty}$ converges strongly to A? Motivate your answer. (3p)

(d) Define for $n = 1, 2, 3, \ldots,$ the operators $T_n \in \mathcal{B}(X, Y)$ by $T_n(f) = f(1/n)$. Does the sequence (T_n) converge in norm? If so, to what? Does the sequence (T_n) converge strongly? If so, to what? (3p)

Do <u>three</u> of the following four problems. They're worth 5p each, and it's your choice which ones to do. (You may not hand in more than three solutions!)

Problem 3: Let Ω denote an equicontinuous subset of $C_{\rm b}(\mathbb{R})$, and let $\Omega_{\rm b}$ denote a bounded equicontinuous subset of $C_{\rm b}(\mathbb{R})$.

(a) Prove that the set $\tilde{\Omega} = \{f^2 \,:\, f \in \Omega\}$ does not need to be equicontinuous.

(b) Is the set $\tilde{\Omega}_{\rm b} = \{f^2 \,:\, f \in \Omega_{\rm b}\}$ necessarily equicontinuous? Give a proof or a counter-example.

Problem 4: Let X denote the set of all continuous functions on the interval $I = [-\pi,\,\pi]$. Equip X with the norm

$$||f|| = \int_{-\pi}^{\pi} |f(y)|\,dy.$$

Consider the operator $T \in \mathcal{B}(X)$ that is defined by

$$[Tf](x) = \int_{-\pi}^{\pi} \sin(x)\,e^{y}\,f(y)\,dy.$$

Calculate the norm of T in $\mathcal{B}(X)$.

Problem 5: Recall that a topological space $(X, \mathcal{T})$ is said to be "Hausdorff" if given any two distinct points $x, y \in X$, there exists disjoint sets $G_x, G_y \in \mathcal{T}$ such that $x \in G_x$ and $y \in G_y$. Prove that in a Hausdorff space, a compact set must necessarily be closed. You may *not* assume that X is metrizable.

Problem 6: Consider the Hilbert space $H = l^2(\mathbb{Z})$. Recall that if $x, y \in H$, with $x = (\dots,\,x_{-1},\,x_0,\,x_1,\,x_2,\dots)$ and $y = (\dots,\,y_{-1},\,y_0,\,y_1,\,y_2,\dots)$, then the inner product in H is defined by

$$(x,\,y) = \sum_{n=-\infty}^{\infty} \bar{x}_n\,y_n.$$

Let A denote the set of all even sequences, in other words

$$A = \{x \in H \,:\, x_n = x_{-n}\}.$$

Consider the vector $x \in H$ for which

$$x_n = \begin{cases} 2^{-n} & \text{when } n \geq 0, \\ 0 & \text{when } n < 0. \end{cases}$$

Compute the number

$$d = \inf_{y \in A} ||x - y||.$$