Applied Analysis (APPM 5440): Final Exam
7.30am – 10.00am, Dec. 20, 2006. Closed books.

Problem 1: The following problems are worth 2p each. No motivation required for (a), (e), or (f). For the rest, please motivate your answer briefly (at most one or two sentences).

(a) State the contraction mapping theorem.

(b) Let H be a Hilbert space, and let Ω be a subset of H. Under what conditions is it necessarily the case that $H = \Omega \oplus \Omega^{\perp}$?

(c) Given a separable Hilbert space H, and an orthonormal sequence $(e_n)_{n=1}^{\infty}$, specify two different conditions that guarantee that $(e_n)_{n=1}^{\infty}$ is a basis for H.

(d) Let H be a Hilbert space, and let $\varphi \in H^*$. Set $N = \ker(\varphi)$. Suppose that $x, y \in N^{\perp}$. Is it necessarily the case that $|(x,\, y)| = ||x||\,||y||$?

(e) State the Arzelà-Ascoli theorem.

(f) Let (X, d) be a complete metric space and let Ω be a subset of X. Is it true that Ω is pre-compact if and only if it is totally bounded?

(g) Let X be a set with a finite number of elements. Is it true that all metrics on X induce the same topology?

(h) Is the function $f(x) = \sqrt{x}\,\sin\frac{1}{x}$ uniformly continuous on the interval $I = (0,\, 1]$?

Solution:

(a,c,e,f) See textboook / lecture notes.

(b) When Ω is a linear subspace that is <u>topologically closed</u>.

(d) Yes. According to the Riesz theorem, there exists a unique $z \in H$ such that $\varphi(x) = (z,\, x)$. Consequently $N^{\perp} = \text{span}(z)$, and so x and y are necessarily parallel.

(g) Yes. In a metrizable topology on a finite set, every singleton set must be open (since for every $x \in X$, we have $\{x\} = B_{\varepsilon}(x)$ where $\varepsilon = \frac{1}{2}\min_{y \in X} d(x, y) > 0$). Therefore such a topology must be the discrete topology.

(h) Yes. To see this, consider the function

$$\bar{f}(x) = \begin{cases} \sqrt{x}\,\sin(1/x) & \text{when } x > 0, \\ 0 & \text{when } x = 0, \end{cases}$$

on the compact set $\bar{I} = [0,\, 1]$. The function $\bar{f}$ is uniformly continuous since it is a continuous function on a compact set, and then f must also be uniformly continuous since it is the restriction of the function $\bar{f}$ to I.

Problem 2: Set $I = [0, 1]$ and consider the Banach space $X = C_{\mathrm{b}}(I)$ (with the usual norm, $||f||_{\mathrm{u}} = \sup_{x \in I} |u(x)|$). Let $Y = \mathbb{R}$ with the standard norm.

(a) Let $A_n \in \mathcal{B}(X, Y)$. Define what it means for the sequence $(A_n)_{n=1}^{\infty}$ to converge strongly in $\mathcal{B}(X, Y)$. (1p)

(b) Let $A_n \in \mathcal{B}(X, Y)$. Define what it means for the sequence $(A_n)_{n=1}^{\infty}$ to converge in norm in $\mathcal{B}(X, Y)$. (1p)

(c) Let $(A_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{B}(X, Y)$ such that for every $\varepsilon > 0$, there exists an integer N such that $||A_n - A_m|| < \varepsilon$ when $m, n \geq N$. Is it necessarily the case that there exists a unique $A \in \mathcal{B}(X, Y)$ such that the sequence $(A_n)_{n=1}^{\infty}$ converges strongly to A? Motivate your answer. (3p)

(d) Define for $n = 1, 2, 3, \ldots$, the operators $T_n \in \mathcal{B}(X, Y)$ by $T_n(f) = f(1/n)$. Does the sequence (T_n) converge in norm? If so, to what? Does the sequence (T_n) converge strongly? If so, to what? (3p)

Solution:

(a) A_n converges strongly to A if, for every $f \in X$, $||A_n\, f - A\, f|| \to 0$ as $n \to \infty$.

(b) A_n converges in norm to A if $||A_n - A|| \to 0$ as $n \to \infty$.

(c) Yes. Since Y is complete, so is $\mathcal{B}(X, Y)$. Therefore the Cauchy sequence (A_n) has a limit point A in the norm topology. Since we know that $\lim_{n\to\infty} ||A_n - A|| = 0$, it follows that for any $f \in X$, $\limsup_{n\to\infty} ||A_n\, f - A\, f|| \leq \limsup_{n\to\infty} ||A_n - A||\, ||f|| = 0$, and so (A_n) converges to A strongly.

(d) Let $T \in \mathcal{B}(X, Y)$ denote the operator defined by $T f = f(0)$. Then $T_n \to T$ strongly, since, for a given $f \in X$, we have

$$T_n\, f = f(1/n) \to f(0) = T\, f$$

due to the continuity of f.

Next let us consider whether (T_n) converges in the norm topology. If it did, then the limit point in the norm topology would have to be the same as the limit point in the strong topology. However, we will prove that $||T_n - T|| \geq 1$, which shows that (T_n) does not converge to T in norm, and hence cannot converge to anything in norm.

To prove that $||T_n - T|| \geq 1$, consider the functions

$$f_n(x) = \begin{cases} n\, x & \text{when } x < 1/n, \\ 1 & \text{when } x \geq 1/n. \end{cases}$$

We see that $f_n \in X$, and that $||f_n|| = 1$. Thus

$$||T_n - T|| = \sup_{||f||=1} ||T_n\, f - T\, f|| \geq ||T_n\, f_n - T\, f_n|| = |f_n(1/n) - f_n(0)| = |1 - 0| = 1.$$

Problem 3: Let Ω denote an equicontinuous subset of $C_{\rm b}(\mathbb{R})$, and let $\Omega_{\rm b}$ denote a bounded equicontinuous subset of $C_{\rm b}(\mathbb{R})$.

(a) Prove that the set $\tilde{\Omega} = \{f^2 \,:\, f \in \Omega\}$ does not need to be equicontinuous.

(b) Is the set $\tilde{\Omega}_{\rm b} = \{f^2 \,:\, f \in \Omega_{\rm b}\}$ necessarily equicontinuous? Give a proof or a counter-example.

Solution:

(a) We will provide an example of an equicontinuous set Ω for which $\tilde{\Omega}$ is not equicontinuous. Consider the functions

$$f_n(x) = \begin{cases} n & \text{when } x \in (-\infty, 0), \\ n + x & \text{when } x \in [0, 1], \\ n + 1 & \text{when } x \in (1, \infty), \end{cases}$$

and set $\Omega = \{f_n \,:\, n = 1,\, 2,\, 3,\, \dots\}$. Since all f_n are Lipschitz continuous with Lipschitz constant 1, Ω is equicontinuous. However, $\tilde{\Omega}$ consists of the functions

$$g_n(x) = \begin{cases} n^2 & \text{when } x \in (-\infty, 0), \\ n^2 + 2\,n\,x + x^2 & \text{when } x \in [0, 1], \\ (n+1)^2 & \text{when } x \in (1, \infty), \end{cases}$$

and these functions are *not* equicontinuous at $x = 0$. To see this, not that given any $\delta \in (0,\, 1)$, we have

$$\sup_{y \in B_\delta(0)} |g_n(y) - g_n(0)| = 2\,n\,\delta + \delta^2.$$

It follows that no matter how small δ is, this quantity can be made arbitrarily large by picking a large n.

(b) Yes. To prove this, suppose that $\Omega_{\rm b}$ is a bounded equicontinuous set. Fix an $x \in \mathbb{R}$, and an $\varepsilon > 0$. We need to prove that there exists a $\delta > 0$ such that $|f(x)^2 - f(y)^2| < \varepsilon$ when $|x - y| < \delta$.

Set $M = \sup_{f \in \Omega_{\rm b}} ||f||$. By assumption, M is finite. Since $\Omega_{\rm b}$ is equicontinuous at x, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/(2M)$ when $|x - y| < \delta$. It then follows that when $|x - y| < \delta$

$$\begin{aligned} |f(x)^2 - f(y)^2| &= |(f(x) - f(y))\,(f(x) + f(y))| \\ &\le |f(x) - f(y)|\,(|f(x)| + |f(y)|) < \frac{\varepsilon}{2M}(M + M) = \varepsilon. \end{aligned}$$

Problem 4: Let X denote the set of all continuous functions on the interval $I = [-\pi, \pi]$. Equip X with the norm

$$||f|| = \int_{-\pi}^{\pi} |f(y)|\, dy.$$

Consider the operator $T \in \mathcal{B}(X)$ that is defined by

$$[T f](x) = \int_{-\pi}^{\pi} \sin(x)\, e^y\, f(y)\, dy.$$

Calculate the norm of T in $\mathcal{B}(X)$.

Solution:

Set $g(x) = \sin(x)$. For a given $f \in X$, we then have

$$||T\, f|| = ||g \int_{-\pi}^{\pi} e^y\, f(y)\, dy|| = ||g|| \left| \int_{-\pi}^{\pi} e^y\, f(y)\, dy \right| = 4 \left| \int_{-\pi}^{\pi} e^y\, f(y)\, dy \right|,$$

since $||g|| = \int_{-\pi}^{\pi} |\sin(x)|\, dx = 4$.

Now

$$||T\, f|| = 4 \left| \int_{-\pi}^{\pi} e^y\, f(y)\, dy \right| \leq 4 \left(\sup_{y \in [-\pi,\pi]} |e^y| \right) \int_{-\pi}^{\pi} |f(y)|\, dy = 4\, e^{\pi}\, ||f||,$$

so $||T|| \leq 4\, e^{\pi}$.

To conversely prove that $||T|| \geq 4\, e^{\pi}$, let us use the functions

$$f_n(x) = \begin{cases} 0 & \text{when } x \in [-\pi, \pi - 1/n], \\ 2\, n^2\, (x - \pi + 1/n) & \text{when } x \in (\pi - 1/n, \pi]. \end{cases}$$

Since $f_n \in X$, and $||f_n|| = 1$, we find that

$$\begin{aligned} ||T|| = \sup_{||f||=1} ||T\, f|| \geq \sup_n ||T\, f_n|| &= \sup_n \left[4 \int_{\pi - 1/n}^{\pi} e^y\, f_n(y)\, dy \right] \\ &\geq \sup_n \left[4 \left(\inf_{x \in [\pi-1/n, \pi]} e^y \right) \int_{\pi-1/n}^{\pi} f_n(y)\, dy \right] = \sup_n \left[4\, e^{\pi - 1/n} \right] = 4\, e^{\pi}. \end{aligned}$$

Problem 5: Recall that a topological space $(X, \mathcal{T})$ is said to be "Hausdorff" if given any two distinct points $x, y \in X$, there exists disjoint sets $G_x, G_y \in \mathcal{T}$ such that $x \in G_x$ and $y \in G_y$. Prove that in a Hausdorff space, a compact set must necessarily be closed. You may *not* assume that X is metrizable.

Solution:

Let K be a compact set in a Hausdorff space X. We will prove that K is closed by proving that $K^{\rm c}$ is open. In turn, to prove that $K^{\rm c}$ is open, we will pick an arbitrary point $x \in K^{\rm c}$ and construct an open set G such that $x \in G \subseteq K^{\rm c}$.

Fix a point $x \in K^{\rm c}$. For each $y \in K$, let H_y and G_x denote disjoint open sets containing y and x, respectively. Then $\{H_y\}_{y \in K}$ is an open cover of K, and since K is compact, there exists a finite subcover $\{H_{y_j}\}_{j=1}^{n}$. Set

$$G = \bigcap_{j=1}^{n} G_{y_j}.$$

That $x \in G$ is obvious since $x \in G_{y_j}$ for each j. That $G \subseteq K^{\rm c}$ is also clear since

$$K \subseteq \bigcup_{j=1}^{n} H_{y_j} \subseteq \bigcup_{j=1}^{n} (G_{y_j})^{\rm c} = \left(\bigcap_{j=1}^{n} G_{y_j}\right)^{\rm c} = G^{\rm c}. \tag{1}$$

Problem 6: Consider the Hilbert space $H = l^2(\mathbb{Z})$. Recall that if $x, y \in H$, with $x = (\dots, x_{-1}, x_0, x_1, x_2, \dots)$ and $y = (\dots, y_{-1}, y_0, y_1, y_2, \dots)$, then the inner product in H is defined by

$$(x, y) = \sum_{n=-\infty}^{\infty} \bar{x}_n \, y_n.$$

Let A denote the set of all even sequences, in other words

$$A = \{x \in H : \ x_n = x_{-n}\}.$$

Consider the vector $x \in H$ for which

$$x_n = \begin{cases} 2^{-n} & \text{when } n \geq 0, \\ 0 & \text{when } n < 0. \end{cases}$$

Compute the number

$$d = \inf_{y \in A} ||x - y||.$$

Solution:

A is a closed linear subspace, so $H = A \oplus A^{\perp}$. It follows that if $x = y + z$ where $y \in A$, and $z \in A^{\perp}$, then

$$d = \inf_{v \in A} ||x - v|| = ||x - y|| = ||z||,$$

so we just need to construct z.

Define $y, z \in H$ by

$$y_n = \tfrac{1}{2}(x_n + x_{-n}), \qquad z_n = \tfrac{1}{2}(x_n - x_{-n}).$$

It is obvious that $x = y + z$, and that $y \in A$. That $z \in A^{\perp}$ follows from a simple calculation (if $v \in A$, then $(v, z) = \frac{1}{2}\sum \bar{v}_n x_n - \frac{1}{2}\sum \bar{v}_n x_{-n} = \frac{1}{2}(v, x) - \frac{1}{2}(v, x) = 0$).

We find that

$$z_n = \begin{cases} -\frac{1}{2}2^{-n} & \text{when } n < 0, \\ 0 & \text{when } n = 0, \\ \frac{1}{2}2^{-n} & \text{when } n > 0, \end{cases}$$

and so

$$||z||^2 = 2\sum_{n=1}^{\infty} \left|\frac{1}{2}2^{-n}\right|^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{8}\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{8}\frac{1}{1 - 1/4} = \frac{1}{6}.$$

It follows that

$$d = ||z|| = \frac{1}{\sqrt{6}}.$$