Homework set 3 — APPM5440, Fall 2006

From the textbook: 1.17, 1.18, 1.20, 1.22.

Problem 1: Let (X, d) be a metric space, and let $\Omega \subseteq X$. Prove that Ω is dense in X if and only if for every $x \in X$, and every $\varepsilon > 0$, there exists a $y \in \Omega$ such that $y \in B_\varepsilon(x)$.

Problem 2: Suppose that $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are Cauchy sequences in a metric space (X, d). Prove that the sequence $(d(x_n, y_n))_{n=1}^\infty$ converges.

Problem 3: In the proof that every metric space has a completion, we did not prove that the function $\tilde{d}$ is indeed a metric. Verify that this is the case.

Optional problem: In the proof that every metric space has a completion, the technique we used to prove that $\tilde{X}$ is complete is called the "Cantor diagonal argument". This is a standard technique. Try to use it to prove that the real numbers are not countable. Hint: Assume that there exists an enumeration $(r^{(n)})_{n=1}^\infty$ of all real numbers in the interval $(0, 1)$. Suppose that each $r^{(n)}$ has a binary number expansion

$$r^{(n)} = 0.b_1^{(n)}\, b_2^{(n)}\, b_3^{(n)} \dots$$

(so that each $b_j^{(n)}$ is either 0 or 1) and use the "diagonal" technique to construct a real number that is not in the sequence. (There is a full solution in the Wikipedia article on Cantor's diagonal argument.)