201A, HW#1, Soln Set.

Ex 1.2.
$$\sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1-x}$$

So

$$\left| \sum_{n=0}^{N} x^n - \frac{1}{1-x} \right| \leq \frac{|x|^{N+1}}{1-|x|}$$

$$\to 0 \quad \text{as } N \to \infty$$

for $|x|<1$. Note that $\frac{|x|^{N+1}}{1-|x|}$ is monotone decreasing.

Therefore, $\forall\ \varepsilon > 0$, pick $\delta > 0$ such that $\frac{\delta^{N+1}}{1-\delta} < \varepsilon$.

1.3 Start from the triangle inequality:

$$d(x,y) + d(y,z) \geq d(x,z)$$

Therefore,

$$d(x,y) \geqslant d(y,z) - d(x,z)$$

By interchanging the roles of x and y, we also get

$$d(x,y) \geqslant d(x,z) - d(y,z)$$

So

$$d(x,y) \geqslant |d(x,z) - d(y,z)|$$

Ex 1.5 Suppose $(X, \|\cdot\|)$ is a normed linear space.

- Prove that $d(x,y) = \|x-y\|$ defines a metric on X.

Proof: Due to the properties of a norm, we immediately have

$d(y,x) = d(x,y) \geq 0$ and $d(x,y)=0$ iff $x=y$

The triangle inequality follows from the triangle inequality for the norm:

$$d(x,y) = \|x-y\| = \|(x-z)+(z-y)\|$$

$$\leq \|x-z\| + \|z-y\|$$

$$\leq d(x,z) + d(z,y)$$

- Prove that $d(x,y) = \dfrac{\|x-y\|}{1+\|x-y\|}$ defines a metriz on X.

Proof: We will prove a more general result.
Suppose $f : [0,\infty) \to [0,\infty)$ has the following properties

(i) $f(0)=0$, $f(x)>0$ for $x>0$

(ii) f is non-decreasing

(iii) f is concave ($\Leftrightarrow$ f' is non-increasing)

Claim Then, for any metrix d on X,

$$d_f(x,y) = f(d(x,y))$$ also defines

a metric on X.
It is easy to check, using calculus, that $f(x) = \frac{x}{1+x}$ satisfies (i)–(iii).

<u>Proof of the claim:</u>
Again the only non-trivial property is the triangle inequality. First note that that f has the property that for any $a, b \geq 0$

$$f(a+b) \leq f(a) + f(b)$$

To see this, use property (iii) as follows:

$$\begin{aligned} f(a+b) &= f(a) + f(a+b) - f(a) \\ &= f(a) + \int_a^{a+b} f'(t)\,dt \\ &= f(a) + \int_0^b f'(a+s)\,ds \\ &\overset{(iii)}{\leq} f(a) + \int_0^b f'(s)\,ds \\ &\leq f(a) + f(b) - f(0) \\ &\overset{(i)}{=} f(a) + f(b) \end{aligned}$$

Now, we can prove the triangle inequality for d_f:

$$\begin{aligned} d_f(x,y) &= f(d(x,y)) \overset{(iii)}{\leq} f(d(x,z) + d(z,y)) \\ &\leq f(d(x,z)) + f(d(z,y)) = d_f(x,z) + d_f(z,y) \end{aligned}$$

1.9 Let (x_n) be a bounded sequence in $\mathbb{R}$.

(a) Prove that for every $\varepsilon > 0$ and every $N \in \mathbb{N}$ $\exists$ $n_1, n_2 \geq N$ s.t.

$$\limsup_n x_n \leq x_{n_1} + \varepsilon$$

$$\liminf_n x_n \geq x_{n_2} - \varepsilon$$

Proof: As the limsup and liminf depend only on the tail of the sequence, WLOG we can assume that $N = 1$ (i.e. if we prove it for $N=1$, we can apply the result to the sequence $y_n = x_{n+N}$).

The second ~~for~~ inequality follows from the first by applying it to the sequence $(-x_n)$ and using that

$$\limsup_n -x_n = -\liminf_n x_n$$

So, let us prove the first inequality. Suppose such n_1 did not exist, then

$$x_{n_1} < \limsup_n x_n - \varepsilon$$

for all n_1, and hence $\limsup_n x_n - \varepsilon$ would be an upper bound for the sequence, implying the contradiction

$$\limsup_n x_n \leq \limsup_n x_n - \varepsilon$$

As x_n is assumed to be bounded this is impossible

(b) Prove that for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ s.t.

$$x_m \leq \limsup_n x_n + \varepsilon \quad \forall m \geq N$$

and $$x_m \geq \liminf_n x_n - \varepsilon \quad \forall m \geq N$$

Proof: Again, the second inequality follows from the first by considering the sequence $(-x_n)$. The first inequality follows immediately from the definition of lim sup:

$$\limsup_n x_n = \lim_{m \to \infty} \sup \{x_k \mid k \geq m\}$$

it follows that $\forall \varepsilon > 0 \ \exists N$ such that

$$\left|\left(\limsup_n x_n\right) - \sup_{k \geq m} x_k\right| < \varepsilon$$

$\forall m \geq N$. A fortiori

$$x_m \leq \sup_{k \geq m} x_k \leq \varepsilon + \limsup_n x_n$$

▨

(e) First, suppose x_n converges, say to x. Then, from part (a) we can find a subsequence x_{n_k} such that

$$\limsup_n x_n \leq x_{n_k} + \varepsilon \quad \forall k$$

As $x_n \to x$, also $x_{n_k} \to x$. Therefore,

$$\limsup_n x_n \leq x + \varepsilon$$

Similarly

$$x - \varepsilon \leq \liminf_n x_n$$

So

$$x - \varepsilon \leq \liminf_n x_n \leq \limsup_n x_n \leq x + \varepsilon$$

As $\varepsilon > 0$ is arbitrary, this implies

$$\limsup_n x_n = \liminf_n x_n = x .$$

Second, suppose $\limsup_n x_n = \liminf_n x_n = L$. Then, from part (b), we see that

$\forall \varepsilon > 0, \ \exists N, \text{ s.t. } \forall m \geq N$

$$L - \varepsilon \leq x_m \leq L + \varepsilon$$

or $$|x_m - L| < \varepsilon.$$

By the definition of convergence of a sequence this means

$$x_m \longrightarrow L$$

$\blacksquare$

1.10. First, we show that if

$$a_n \le b_n \quad \text{for all } n \in \mathbb{N}$$

then

$$\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n .$$

<u>Proof</u>: Given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$b_n \le \limsup_{n\to\infty} b_n + \varepsilon \qquad \forall\, n \ge N,$$

and there is an $n_1 \ge N$ such that

$$\limsup_{n\to\infty} a_n \le a_{n_1} + \varepsilon .$$

Hence

$$\begin{aligned} \limsup_{n\to\infty} a_n &\le a_{n_1} + \varepsilon \\ &\le b_{n_1} + \varepsilon \\ &\le \limsup_{n\to\infty} b_n + 2\varepsilon . \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the result follows. $\square$

For every $\alpha \in A$, we have

$$\inf_{\beta \in A} x_{n,\beta} \leq x_{n,\alpha}.$$

Therefore, by the previous result,

$$\limsup_{n\to\infty} \left(\inf_{\beta \in A} x_{n,\beta}\right) \leq \limsup_{n\to\infty} x_{n,\alpha}.$$

It follows that $\limsup_{n\to\infty} \left(\inf_{\beta\in A} x_{n,\beta}\right)$ is a lower bound of the set $\{\limsup_{n\to\infty} x_{n,\alpha} \mid \alpha \in A\}$, so

$$\limsup_{n\to\infty} \left(\inf_{\beta\in A} x_{n,\beta}\right) \leq \inf_{\alpha\in A} \left(\limsup_{n\to\infty} x_{n,\alpha}\right).$$

The corresponding result for the liminf of sup follows by application of this result to $\{-x_{n,\alpha}\}$.

Example of strict inequality

Let $A = \mathbb{N}$, and define

$$x_{n,m} = \begin{cases} 0 & n \le m \\ 1 & n > m \end{cases}$$

Then

$$\limsup_{n\to\infty} x_{n,m} = 1 \quad \text{for all } m \in \mathbb{N}$$

$$\inf_{m\in\mathbb{N}} x_{n,m} = 0 \quad \text{for all } n \in \mathbb{N}$$

So

$$\limsup_{n\to\infty} \left(\inf_{m\in\mathbb{N}} x_{n,m} \right) = 0$$

$$\inf_{m\in\mathbb{N}} \left(\limsup_{n\to\infty} x_{n,m} \right) = 1 .$$

1.12 Let $f: X \to Y$, $g: Y \to Z$ be continuous functions. Show that $h: X \to Z$, $h = g \circ f$ is also continuous.

<u>Proof</u> The shortest proof is by applying Proposition 1.46. So, we need to argue that for any $G \subset Z$, open, $h^{-1}(G)$ is open. This follows by two more applications of Proposition 1.46:

$$h^{-1}(G) = f^{-1}(g^{-1}(G))$$

which is open by the continuity of g and f.

$\blacksquare$

1.16 First, we show that (X = metric space)

$$d(\cdot, E) : X \longrightarrow \mathbb{R}$$

is continuous. If $x_n \to x$, then

$$d(x,E) = \inf_{y \in E} d(x,y)$$

$$= \inf_{y \in E} \lim_{n \to \infty} d(x_n, y)$$

$$\geq \limsup_{n \to \infty} \left[\inf_{y \in E} d(x_n, y) \right]$$

$$\geq \limsup_{n \to \infty} d(x_n, E).$$

So $d(\cdot, E)$ is upper semicontinuous.

To prove that $d(\cdot, E)$ is also lower semicontinuous (and hence continuous), we let $\varepsilon > 0$. If $x_n \to x$, there exists $y_n \in E$ such that

$$d(x_n, y_n) < d(x_n, E) + \varepsilon$$

$$\Rightarrow \quad d(x, y_n) \leq d(x, x_n) + d(x_n, y_n)$$

$$\Rightarrow \quad d(x, y_n) \leq d(x, x_n) + d(x_n, E) + \varepsilon$$

Letting $n \to \infty$, we get

$$d(x,E) \leq \liminf_{n\to\infty} d(x_n,E) + \varepsilon.$$

Since this inequality holds for all $\varepsilon > 0$, we conclude that $d(\cdot,E)$ is lower semicontinuous.

If F is closed, then $d(x,F)=0$ if and only if $x \in F$: if $d(x,F)=0$ then there exist $x_n \in F$ such that $d(x,x_n) \to d(x,F)=0$; hence $x_n \to x$ and $x \in F$ since F is closed.

Since $F \cap G^c = \phi$ and F, G^c are closed, we have

$$d(x,F) + d(x,G^c) \neq 0 \quad \forall x \in X.$$

Hence

$$f(x) = \frac{d(x,G^c)}{d(x,F) + d(x,G^c)}$$

is continuous, and

$$f(x) = \begin{cases} 0 & x \notin G \\ 1 & x \in F \end{cases}.$$