201A HW#2 Solution Set

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HN# 1.17

- If Y is not closed, then $\overline{Y} \neq Y$, i.e. $\exists x \in \overline{Y} \setminus Y$ and a sequence (y_n), $y_n \in Y$, $\forall n \geq 1$, $y_n \to y$. Therefore, Y is not complete.

- If Y is not complete, $\exists$ Cauchy sequence (y_n), $y_n \in Y$, $\forall n \geq 1$, but y_n does not converge to a limit in Y. However, $X \supseteq Y$ is complete, therefore $\exists\, x \in X$ s.t. $y_n \to x$. Therefore, Y is not closed

HN# 1.18

Pick n_1 s.t. $d(x_k, x_l) \leq \varepsilon_1 \quad \forall k, l \geq n_1$

Then, pick $n_2 > n_1$ s.t $d(x_k, x_l) \leq \varepsilon_2 \quad \forall k, l \geq n_2$

$\vdots$ repeat at ∞

pick $n_{k+1} > n_k$ s.t. $d(x_n, x_m) \leq \varepsilon_{k+1} \; \forall m, n \geq n_{k+1}$

$\vdots$

Such n_{k+1} always exists because (x_n) is Cauchy

HN #1.20

- Suppose $\sum_n x_n$ is absolutely convergent for $x_n \in X$, a Banach space.
  Define
  $$S_N = \sum_{n=1}^{N} x_n \quad , \quad s_N = \sum_{n=1}^{N} \|x_n\|$$

Then, for $N \geq M$
$$\| S_N - S_M \| = \left\| \sum_{n=M+1}^{N} x_n \right\|$$
$$\leq \sum_{n=M+1}^{N} \|x_n\|$$
$$= | s_N - s_M |$$

(s_n) is Cauchy in $\mathbb{R}$ $\Leftrightarrow$ $\left(\sum_n x_n \text{ converges absolutely}\right)$
$\Rightarrow \forall \varepsilon > 0 \;\exists M$ such that $\|s_N - s_M\| < \varepsilon$
$\Rightarrow \| S_N - S_M \| < \varepsilon$

$\Rightarrow (S_N)$ Cauchy in X

$\Rightarrow$ converges

- Next suppose $(X, \|\cdot\|)$ is normed linear space such that every absolutely convergent series converges. Let (x_n) be Cauchy sequence in X. We need to show that x_n converges in X.

We will use the property that if a

Cauchy sequence has a convergent subsequence, then it converges.

From 1.18 we know that we can pick a subsequence (y_n) of (x_n) such that

$$d(y_n, y_m) \leq 2^{-m} \quad \forall n \geq m$$

We will show that (y_n) converges, by showing that y_n is the partial sum of an absolutely convergent series. Define $y_0 = 0$. Then

$$y_n = \sum_{n=1}^{n} (y_n - y_{n-1})$$

and $$\sum_{n=1}^{\infty} \| y_n - y_{n-1} \| \leq \sum_{n=1}^{\infty} 2^{-(n-1)} < +\infty$$

$\boxtimes$

HW #1.23 For $x_n \to x$

- If $\liminf_n f(x_n) < M$, then $\exists N, \exists \varepsilon > 0$

such that $f(x_n) \leq M - \varepsilon \quad \forall \quad n \geq N$

and $f(x) < M$ as well (lsc of f)

$\Rightarrow \liminf_n f_M(x_n) = \liminf_n f(x_n) \geq f(x) = \min(f(x), M) = f_M(x)$

• If $\liminf f(x_n) \geq M$, then

$$\liminf_n f_M(x_n) = \liminf_n f(x_n)$$

$$\geq f(x) \geq \min(f(x), M) = f_M(x)$$

$\blacksquare$

HW#1.25

As f is coercive, $\exists\, M > 0$ such that

$$f(x) > f(0) \quad \forall\, x,\ \|x\| \geq M.$$

Then, $\exists$ sequence (x_n), $x_n \in [-M,M]^n$

such that $\inf_{x \in \mathbb{R}^n} f(x) = \lim_n f(x_n)$

By compactness of $[-M,M]^n$, $\exists$ subsequence of (x_n) say (y_n), such that $y_n \to y$.

By l.s.continuity of f, we then have

$$\inf_x f(x) = \lim_n f(y_n) = \liminf_n f(y_n) \geq f(y)$$

$\Rightarrow$ f attains its infimum in $y \in [-M,M]^n$.
In particular $\inf f \in \mathbb{R}$, i.e. f is bounded below. $\blacksquare$

HW#1.26

No. Consider $p(x,y) = x^2 + (1-xy)^2$

Clearly $p(x,y) \geq 0$. Suppose $\exists$ (x_0, y_0) s.t. $p(x_0,y_0) = 0$, then each of the two terms should vanish, which is impossible.

HW#1.27

Suppose that such a sequence (x_n) did not converge to x. By definition $\exists\ \varepsilon > 0$ such that $\forall N$ $\exists\ n \geq N$ with $d(x, x_n) \geq \varepsilon$. In particular there is subsequence (y_n) with $d(y_n, x) \geq \varepsilon$. As the metric space is compact, y_n has a convergent subsequence $z_n \to z$, which is also a subsequence of x_n, but

$$d(x,z) = \lim_n d(x, z_n) \geq \varepsilon$$

So $x \neq z$, contradicting the assumption that every convergent subsequence of (x_n) converges to x.

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