Applied Analysis (APPM 5440): Midterm 2

5.30pm – 6.45pm, Oct. 30, 2006. Closed books.

Problem 1: Let (X, d) be a metric space.

(a) Define what it means for a subset Ω of X to be *compact.* (2p)

(b) Let $(x_n)_{n=1}^{\infty}$ be a sequence in Ω. Suppose that there exists an $\varepsilon > 0$, such that if $n \neq m$, then $d(x_n, x_m) \geq \varepsilon$. Prove directly from the definition given in (a) that Ω cannot be compact. (2p)

(*) *If you have time left, this problem could earn you 1p extra:* Give a definition of compactness that is equivalent to, but different from, the one you gave in (a), and redo (b) using this definition.

Problem 2: State the Grönwall inequality. (2p)

Problem 3: Consider the equation

$$u(x) + \int_0^x u(y)^3 \, dy = 1. \tag{1}$$

Prove that for some positive δ, equation (1) has a unique solution in $C_{\mathrm{b}}([0, \delta])$. (5p)

Hint: You may want to work with a bounded subspace of $C_{\mathrm{b}}([0, \delta])$, rather than the space itself.

Problem 4: Set $I = (0, 1)$, $X = C_{\mathrm{b}}(I)$, and

$$\Omega = \{f \in X : \ f \text{ has compact support in } I \text{ and } ||f|| \leq 1\}.$$

(a) Prove that Ω is not closed in X. (2p)

(b) Describe $\bar{\Omega}$, the closure of Ω in X. (1p)

(c) Is $\bar{\Omega}$ a compact set in X? Prove that your answer is correct. (3p)

Problem 5: Let f denote a fixed function in $C_{\mathrm{b}}(\mathbb{R})$. Set $\Omega = \{f_n\}_{n=1}^{\infty}$ where $f_n(x) = f(x - n)$. Is Ω is necessarily equicontinuous? Motivate your answer. (3p)