Homework set 12 — APPM5440 — Fall 2012

Problem 1: Let X denote the linear space of polynomials of degree 2 or less on $I = [0, 1]$. For $f \in X$, set $||f|| = \sup_{x\in I} |f(x)|$. For $f \in X$, define

$$\varphi_1(f) = \int_0^1 f(x)\,dx, \quad \varphi_2(f) = f(0), \quad \varphi_3(f) = f'(1/2), \quad \varphi_4(f) = f'(1/3).$$

Prove that $\varphi_j \in X^*$ for $j = 1, 2, 3, 4$. Prove that $\{\varphi_1, \varphi_2, \varphi_3\}$ forms a basis for X^*. Prove that $\{\varphi_1, \varphi_2, \varphi_4\}$ does not form a basis for X^*.

Hint: Any $f \in X$ can be written $f(x) = a + b\,x + c\,x^2$ for unique a, b, and c.

Problem 2: Let $X = \ell^2$. Recall from class that every $\varphi \in X^*$ is of the form $\varphi(x) = \sum x_n\,y_n$ for some $y \in X$. Set $D = \{x \in \ell^2 : ||x|| = 1\}$. Prove that the weak closure of D is the closed unit ball in ℓ^2. (Hint: To prove that the closed unit ball is contained in the weak closure of D, you can for any element x such that $||x|| < 1$ explicitly construct a sequence $(x^{(n)})_{n=1}^{\infty} \subset D$ that weakly converges to x, such that $||x^{(n)}|| = 1$.)

Set $Y = \ell^3$. What is Y^*? Prove that the weak closure of the surface of the unit ball in ℓ^3 is the closed unit ball in ℓ^3.

Problem 3: Let X be a normed linear space, let M be a closed subspace, and let $\hat{x}$ be an element not contained in M. Set

$$d = \text{dist}(M, \hat{x}) = \inf_{y\in M} ||y - \hat{x}||.$$

Prove that $d > 0$. Prove that there exists an element $\varphi \in X^*$ such that $\varphi(\hat{x}) = 1$, $\varphi(y) = 0$ for $y \in M$, and $||\varphi|| = 1/d$.

Hint: Set $Z = \text{Span}(M, \hat{x})$. Prove that any $z \in Z$ can be written $z = y + \alpha\,\hat{x}$ for a unique $\alpha \in \mathbb{R}$ and a unique vector $y \in M$. Define ψ as a suitable functional on Z, and then extend it to X using the Hahn-Banach theorem.

Problem 4: Let X be a normed linear space with a linear subspace M. Prove that the weak closure of M equals the closure of M in the norm topology. *Hint:* Use Problem 3.

Problem 5: Prove that the following statements follow from the Hahn-Banach theorem:

(a) For any $x \in X$, there is a $\varphi \in X^*$ such that $||\varphi|| = 1$ and $\varphi(x) = ||x||$.

(b) For any $x \in X$, $||x|| = \sup_{||\varphi||=1} |\varphi(x)|$.

(c) If $x, y \in X$ and $x \neq y$, there is a $\varphi \in X^*$ such that $\varphi(x) \neq \varphi(y)$.

(d) For $x \in X$, define $F_x \in X^{**}$ by setting $F_x(\varphi) = \varphi(x)$.
Prove that the map $x \mapsto F_x$ is a linear isometry from X to X^{**}.

Note that we did this in class — try to repeat the proof without looking at the notes! (We did not prove that the map $x \mapsto F_x$ is linear, you need to do this yourself.)