APPM5440 — Applied Analysis: Section exam 1
17:15 – 18:30, Sep. 25, 2012. Closed books.

Please motivate all answers unless the problem explicitly states otherwise.

Problem 1: (24 points) The following questions are worth 8 points each.

(a) Specify which of the following could potentially be the set C of cluster points of a sequence $(x_n)_{n=1}^{\infty}$ of real numbers. Any negative answer needs a brief motivation.
(1) $C = [0, 1]$.
(2) $C = (0, 1)$.
(3) $C = [0, \infty)$.
(4) $C = \mathbb{Q}$ (the set of rational numbers).
(Recall that given a sequence (x_n), its set of cluster points is defined as the set of limit points of sub-sequences of (x_n).)

(b) Let (X, d) be a metric space. State the definition of the completion of (X, d).

(c) Which of the following statements are true (no motivations required):
(1) If $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers, then $\limsup_{n\to\infty} x_n$ exists.
(2) If (X, d) is a compact metric space, and $(x_n)_{n=1}^{\infty}$ is a sequence in X with the property that every convergent subsequence has the same limit x, then $x_n \to x$.
(3) Every compact subset of a metric space is necessarily closed.
(4) If (X, d) is a compact metric space and $f : X \to (0, 1)$ is continuous, then the function $g(x) = 1/(1 - f(x))$ is bounded on X.
(5) Let X be a normed linear space, and the B denote the unit ball around the origin. Then B is necessarily totally bounded.

Solution:

(a) Only the set in (1) is possible. (Recall from the homework that C must be closed. The set $[0, \infty)$ is a little bit tricky since it is not closed in the set of *extended* real numbers. So while $C = [0, \infty]$ is possible, $C = [0, \infty)$ is not. This subquestion was graded generously. Note that a set being infinite or uncountable is unproblematic. Consider, e.g., the case where (x_n) is an enumeration of the positive rational numbers. Then $C = [0, \infty]$.)

(b) See text.

(c) (1) is true.
(2) is true (see homework).
(3) is true.
(4) is true since a continuous function on a compact set attains its max.
(Alternatively, note that $f(X)$ must be a closed subset of $(0, 1)$.)
(5) False.

Problem 2: (24 points) Suppose that (X_1, d_1), (X_2, d_2), and (X_3, d_3) are metric spaces, and that $f: X_1 \to X_2$ and $g: X_2 \to X_3$ are continuous. Prove that the composition $h = g \circ f$ defined by

$$h:\ X_1 \to X_3:\ x \mapsto g(f(x))$$

is continuous. State explicitly which definition of continuity you use in your proof.

Solution:

Definition: A function f is continuous if the pre-image of any open set is open.

Let G be an open subset of X_3.

Since g is continuous, $g^{-1}(G)$ is open in X_2.

Since f is continuous, $f^{-1}(g^{-1}(G))$ is open in X_1.

Since $f^{-1}(g^{-1}(G)) = h^{-1}(G)$, this shows that h is continuous.

Problem 3: (24 points) Set $I = [-1,1]$ and let X denote the set of real valued continuous functions on I. For $f \in X$, define the norm

$$||f|| = \int_{-1}^{1} |f(x)|\, dx.$$

Show that X is not a Banach space with respect to this norm.

Solution:

Define $f_n \in X$ via

$$f_n(x) = \begin{cases} -1 & -1 \le x < -1/n \\ nx & -1/n \le x < 1/n \\ 1 & 1/n \le x < 1 \end{cases}$$

The sequence (f_n) is Cauchy. To prove this, suppose that $N \le m \ne n$. Then

$$||f_n - f_m|| = \int_{-1}^{1} |f_n(x) - f_m(x)|\, dx = \int_{-1/N}^{1/N} |f_n(x) - f_m(x)|\, dx \le \int_{-1/N}^{1/N} dx \le 2/N.$$

It remains to show that (f_n) cannot converge to any function $g \in X$. Fix $g \in X$. Then

$$||f_n - g|| = \int_{-1}^{1} |f_n(x) - g(x)|\, dx = A_n + B_n + C_n$$

where

$$A_n = \int_{-1}^{-1/n} |g(x) + 1|\, dx, \qquad B_n = \int_{-1/n}^{1/n} |f_n(x) - g(x)|\, dx, \qquad C_n = \int_{1/n}^{1} |g(x) - 1|\, dx,$$

Set $M = \sup_{x \in I} |g(x)|$. Note that M is finite since I is compact. Since $|f_n(x) - g(x)| \le 1 + M$ it then follows that $B_n \le 2M/n$ and so $\lim_{n\to 0} B_n = 0$. Then

$$\lim_{n\to\infty} ||f_n - g|| = \int_{-1}^{0} |g(x) + 1|\, dx + \int_{0}^{1} |g(x) - 1|\, dx.$$

Since g is continuous, at least one of the two terms must be non-zero. [1]

[1] For full marks on this problem, the last assertion did not need to get proven. But for the curious, note that this follows from elementary analysis. Suppose $g(0) \ne 1$. Set $\varepsilon = |1 - g(0)|/2$. Then pick $\delta > 0$ such that $|g(x) - g(0)| < \varepsilon$ for $|x| \le \delta$. Then $\int_0^1 |g(x) - 1|\, dx \ge \int_0^\delta |g(x) - 1|\, dx = \delta\varepsilon > 0$. If $g(0) = 1$, then you can analogously prove that $\int_{-1}^{0} |g(x) + 1|\, dx > 0$.

Problem 4: (28 points) Let X denote the set of sequences of real numbers $x = (x_1,\, x_2,\, x_3,\, \dots)$ such that $\sum_{n=1}^{\infty} x_n^2 < \infty$, and define for $x \in X$ the norm $||x|| = \big(\sum_{n=1}^{\infty} x_n^2\big)^{1/2}$. Consider the following four subsets of X:

- Let d be a positive integer d and set $A = \{x = (x_1,\, x_2,\, \dots,\, x_d,\, 0,\, 0,\, \dots) : \sum_{n=1}^{d} x_n^2 \leq 1\}$.
- $B = \{x = (x_1,\, x_2,\, x_3,\, \dots) : \ \sum_{n=1}^{\infty} n^2 x_n^2 \leq 1\}$.
- $C = \{x = (x_1,\, x_2,\, x_3,\, \dots) : \ \sum_{n=1}^{\infty} x_n^2 \leq 1\}$.
- $D = \{x = (x_1,\, x_2,\, x_3,\, \dots) : \ \sum_{n=1}^{\infty} |x_n| = 1\}$.

Which of the sets A, B, C, and D are compact?

Solution:

Set A is compact. It is isomorphic with the closed unit ball in $\mathbb{R}^d$, which is compact by the Heine-Borel theorem.

Set B is compact. We first prove that B is totally bounded. Pick $\varepsilon > 0$. Pick N such that $N > 2/\varepsilon$. Then for $x \in X$, set

$$P_N x = (x_1, x_2, \dots,\, x_N,\, 0,\, 0,\, \dots).$$

For $x \in B$, we then have $||x - P_N x||^2 = \displaystyle\sum_{n=N+1}^{\infty} x_n^2 \leq \frac{1}{N^2} \sum_{n=N+1}^{\infty} n^2\, x_n^2 \leq \frac{1}{N^2} \leq \frac{\varepsilon^2}{4}$.

Observe that $P_N B$ is compact by the Heine-Borel theorem. Let $\{B_{\varepsilon/2}(x^{(j)})\}_{j=1}^{J}$ denote a finite $\varepsilon/2$-cover of $P_N B$. Then for any $x \in B$, we know that $||x^{(j)} - P_N x|| < \varepsilon/2$ for some $x^{(j)}$, and then

$$||x - x^{(j)}|| \leq ||x - P_N x|| + ||P_N x - x^{(j)}|| < \varepsilon/2 + \varepsilon/2.$$

Therefore $\{B_\varepsilon(x^{(j)})\}_{j=1}^{J}$ is an ε-cover of B.

Next we show that B is closed. Suppose $(x^{(j)})_{j=1}^{\infty}$ is a Cauchy sequence in B. Since X is complete, there is an $x \in X$ such that $x^{(j)} \to x$. This in particular implies that $\lim_{j\to\infty} x_n^{(j)} = x_n$ for every j. We then find

$$\sum_{n=1}^{\infty} n^2 x_n^2 = \sup_N \lim_{j\to\infty} \sum_{n=1}^{N} n^2 (x_n^{(j)})^2 \leq \liminf_{j\to\infty} \sup_N \sum_{n=1}^{N} n^2 (x_n^{(j)})^2 \leq \liminf_{j\to\infty} \sum_{n=1}^{\infty} n^2 (x_n^{(j)})^2 \leq 1.$$

C is not compact. Consider the vectors

$$e^{(1)} = (1, 0, 0, \dots) \qquad e^{(2)} = (0, 1, 0, 0, \dots) \qquad e^{(3)} = (0, 0, 1, 0, 0, \dots). \tag{1}$$

We find that $e^{(j)} \in C$ for every j. Since $||e^{(j)} - e^{(k)}|| = \sqrt{2}$ whenever $j \neq k$, the sequence $(e^{(j)})_{j=1}^{\infty}$ cannot have a convergent subsequence. (Note that C is closed, though.)

D is not compact. The vectors $e^{(j)}$ defined in (1) all belong to D, so this counter-example works for D as well.

(For the curious, note that in addition to not being totally bounded, the set D is in fact not closed either. To show this, set $x^{(j)} = (1,\, 1/2,\, 1/3,\, 1/4,\, \dots,\, 1/j,\, 0,\, 0,\, 0,\, \dots)$, set $\beta_j = \sum_{n=1}^{j} \frac{1}{n}$, and set $y^{(j)} = \frac{1}{\beta_j} x^{(j)}$. Then $y^{(j)} \in D$ for every j. But $y^{(j)} \to 0$ in X (since $(x^{(j)})$ is a bounded sequence in X and $\beta_j \to \infty$), and $0 \notin D$, so D cannot be closed.)