APPM5440 — Applied Analysis: Section exam 2 — Solutions

17:15 – 18:30, Oct. 30, 2012. Closed books.

Problem 3: (20p) Define for $n = 1, 2, 3, \ldots$ the function $f_n : \mathbb{R} \to \mathbb{R}$ via

$$f_n(x) = e^{-n(x-n)^2}.$$

Let N be a fixed positive integer. In the table below, mark each box corresponding with a true statement with the letter "T". No motivations required.

Solution:

	Ω is equicont. for every $x \in I$	Ω is uniformly equicont. on I	Ω is closed in $C(I)$	Ω is pre-compact in $C(I)$
$\Omega = \{f_n\}_{n=1}^{N}$ and $I = \mathbb{R}$	T	T	T	T
$\Omega = \{f_n\}_{n=1}^{\infty}$ and $I = \mathbb{R}$	T		T	
$\Omega = \{f_n\}_{n=1}^{N}$ and $I = [-N, N]$	T	T	T	T
$\Omega = \{f_n\}_{n=1}^{\infty}$ and $I = [-N, N]$	T	T		T

Some comments:

- $\Omega = \{f_n\}_{n=1}^{N}$ and $I = \mathbb{R}$.

  Recall that any finite set of continuous functions is necessarily equicontinuous. Further, note that $\sup_n \sup_{x\in I} |f_n'(x)| \leq C\sqrt{N}$ (where, I think, $C = \sqrt{2}e^{-1/2}$) so the set is uniformly Lipschitz, and therefore uniformly equicontinuous. Ω is closed since it consists of a finite set of points. It is pre-compact since it is finite (and therefore obviously totally bounded).

- $\Omega = \{f_n\}_{n=1}^{\infty}$ and $I = \mathbb{R}$.

  The sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to zero on any interval $[a, b]$; in consequence, the set Ω is equicontinuous on any fixed x. However, for any $\delta > 0$, you can always find an n such that $|f_n(n) - f_n(n + \delta)| \geq 1/2$, so the sequence is not uniformly equicontinuous. Ω is closed since it consists of a set of well-separated points ($||f_n - f_m|| \geq 1/2$ for any $m \neq n$) so there are no accumulation points. The set is not pre-compact since it is not totally bounded (there exist no finite cover of balls with radius 1/3, for instace).

- $\Omega = \{f_n\}_{n=1}^{N}$ and $I = [-N, N]$.

  Recall that any finite set of continuous functions is necessarily equicontinuous. It is uniformly equicontinuous since I is compact. Ω is closed since it consists of a finite set of points. It is pre-compact since it is finite (and therefore obviously totally bounded).

- $\Omega = \{f_n\}_{n=1}^{\infty}$ and $I = [-N, N]$.

  Ω is uniformly equicontinuous since $\sup_n \sup_{x\in I} |f_n'(x)|$ is finite. Ω is not closed since $f_n \to 0$ uniformly, but $0 \notin \Omega$. Ω is pre-compact by the Arzela theorem (equicontinuous and bounded, while I is compact).

Problem 4: (30p) Set $I = [0, 1]$.

(a) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $C(I)$ such that $\mathrm{Lip}(f_n) \leq 1$. Prove that if $(f_n)_{n=1}^{\infty}$ converges uniformly to a function f, then $\mathrm{Lip}(f) \leq 1$.

(b) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $C(I)$ such that $\mathrm{Lip}(f_n) \leq 1$. Does (f_n) necessarily have a convergent subsequence? Please offer a proof or a counter-example.

(c) Set $\Omega = \{f \in C(I) : \mathrm{Lip}(f) \leq 1 \text{ and } f(0) = 0\}$ Is the set Ω closed? Compact? Pre-compact?

(d) Is the set $\Omega = \{f \in C(I) : ||f|| \leq 1 \text{ and } \mathrm{Lip}(f) \leq 1\}$ dense in the unit ball of $C(I)$?

Solution:

(a)

$$\begin{aligned}\mathrm{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} &= \sup_{x \neq y} \lim_{n \to \infty} \frac{|f_n(x) - f_n(y)|}{|x - y|} \\ &\leq \liminf_{n \to \infty} \sup_{x \neq y} \frac{|f_n(x) - f_n(y)|}{|x - y|} = \liminf_{n \to \infty} \mathrm{Lip}(f_n) \leq 1.\end{aligned}$$

(b) No, consider $f_n(x) = n$ (constant functions). Note that while the sequence *is* equicontinuous, the AA theorem does not apply since we are not assured it is bounded.

(c) Ω is closed. To prove this, suppose $f_n \to f$ in $C(I)$ and $f_n \in \Omega$. By the result in (a), we know that $\mathrm{Lip}(f) \leq 1$. Since uniform convergence implies point-wise convergence, we find $f(0) = \lim_{n \to \infty} f_n(0) = 0$. Consequently $f \in \Omega$.

Ω is bounded since if $f \in \Omega$, then

$$|f(x)| = |f(0) + (f(x) - f(0))| \leq |f(0)| + |f(x) - f(0)| \leq |f(0)| + \mathrm{Lip}(f)|x - 0| \leq 0 + 1 \cdot 1 = 1.$$

Since I is compact, and Ω is bounded and equicontinuous, the AA-theorem applies so we know that Ω is pre-compact. Since Ω is also closed, it is compact.

(d) No. Consider the function $f(1) = 1 - 2x$. If $||f - g|| \leq 1/3$, then we know that

$$\begin{aligned}\mathrm{Lip}(g) \geq \frac{|g(1) - g(0)|}{|1 - 0|} &= |(g(1) - f(1)) + (f(1) - f(0)) + (f(0) - g(0))| \\ &\geq |f(1) - f(0)| - |g(1) - f(1)| - |g(0) - f(0)| \geq 2 - 1/3 - 1/3 = 4/3.\end{aligned}$$

So $g \notin \Omega$.

Problem 5: (20p) Let $f = f(x, y)$ be a continuous bounded real-valued function on $\mathbb{R}^2$, and let $g = g(x)$ be a continuous real-valued function on $\mathbb{R}$ such that $||g||_{\mathrm{u}} \leq 1$. Now consider for a positive number δ the equation

$$\begin{cases} u_1(x) = \int_0^\delta f(x,y)\,\big(u_2(y)\big)^2\,dy + g(x), \\ u_2(x) = \frac{1}{3}u_1(x) + \frac{1}{3}\big(u_2(x)\big)^2. \end{cases} \tag{1}$$

Show that for δ small enough, the equation (1) is guaranteed to have a unique solution pair (u_1, u_2) of continuous functions on $[0, \delta]$ such that $||u_2||_{\mathrm{u}} \leq 1$. What can you say about $||u_1||_{\mathrm{u}}$?

Solution 1: Inserting the first equation into the second we find that u_2 must satisfy $u_2 = T(u_2)$ where

$$[T(u_2)](x) = \frac{1}{3}\int_0^\delta f(x,y)\big(u_2(y)\big)^2\,dy + \frac{1}{3}g(x) + \frac{1}{3}\big(u_2(x)\big)^2.$$

Set $M = ||f||_{\mathrm{u}}$. We know that M is finite since f is bounded.

Set $I = [0, \delta]$ and $\Omega = \{v \in C(I) : ||v||_{\mathrm{u}} \leq 1\}$. We will show that T is a contraction on Ω if δ is small enough. Since Ω is a closed metric space, the CMT will then assure us a unique solution.

Verify that T maps Ω to Ω: Suppose $u_2 \in \Omega$. Then

$$||T(u_2)|| \leq \frac{1}{3}M\delta||u_2||^2 + \frac{1}{3} + \frac{1}{3}||u_2||^2 \leq \frac{1}{3}M\delta + \frac{1}{3} + \frac{1}{3}.$$

We see that $T(u_2) \in \Omega$ if $M\delta \leq 1$.

Verify that T is a contraction: Suppose $u_2, v_2 \in \Omega$. Then

$$\begin{aligned} ||T(u_2) - T(v_2)|| \leq \frac{1}{3}M\delta\,||u_2^2 - v_2^2|| + \frac{1}{3}\,||u_2^2 - v_2^2|| &= \frac{1}{3}(M\delta + 1)\,||u_2^2 - v_2^2|| \\ &\leq \frac{1}{3}(M\delta + 1)(||u_2|| + ||v_2||)||u_2 - v_2|| \leq \frac{2}{3}(M\delta + 1)||u_2 - v_2||. \end{aligned}$$

We see that T is a contraction if $M\delta < 1/2$.

We have shown that T is a contraction on Ω if $M\delta < 1/2$.

As for the bound on u_1, simply use the first equation to find

$$||u_1|| \leq M\delta||u_2||^2 + ||g|| \leq M\delta + 1 \leq 3/2.$$

Solution 2: We can write (1) as the fixed point problem $u = T(u)$, where T is an operator on the set of pairs of continuous functions on the set $I = [0, \delta]$:

$$T\left(\left[\begin{array}{c} u_1 \\ u_2 \end{array}\right]\right)(x) = \left[\begin{array}{c} \int_0^\delta f(x,y)\,\big(u_2(y)\big)^2\,dy + g(x) \\ \frac{1}{3}u_1(x) + \frac{1}{3}\big(u_2(x)\big)^2 \end{array}\right].$$

Set

$$\Omega = \{\left[\begin{array}{c} u_1 \\ u_2 \end{array}\right] \,:\, ||u_1||_{\mathrm{u}} \leq C \text{ and } ||u_2||_{\mathrm{u}} \leq 1\}$$

for a suitably chosen C. We equip Ω with the norm

$$|||\left[\begin{array}{c} u_1 \\ u_2 \end{array}\right]||| = ||u_1||_{\mathrm{u}} + ||u_2||_{\mathrm{u}}.$$

Let us show that T is a contraction on Ω if C and δ are chosen appropriately. Set $M = ||f||_{\mathrm{u}}$.

First make sure that $T(\Omega) \subseteq \Omega$. We find

$$\begin{aligned} ||T_1(u)||_{\mathrm{u}} \leq& M\delta ||u_2||^2 + ||g|| \leq M\delta + 1, \\ ||T_2(u)||_{\mathrm{u}} \leq& \frac{1}{3}||u_1|| + \frac{1}{3}||u_2||^2 \leq \frac{1}{3}C + \frac{1}{3}. \end{aligned}$$

This leads to the conditions that $C \leq 2$ and $M\delta + 1 \leq C$.

Now check the contraction property:

$$\begin{aligned} |||T(u) - T(v)||| =& ||T_1(u) - T_1(v)||_{\mathrm{u}} + ||T_2(u) - T_2(v)||_{\mathrm{u}} \\ \leq& M\delta\, ||u_2^2 - v_2^2|| + \frac{1}{3}||u_1 - v_1|| + \frac{1}{3}\, ||u_2^2 - v_2^2|| \\ =& (M\delta + \frac{1}{3})\, ||u_2^2 - v_2^2|| + \frac{1}{3}||u_1 - v_1|| \\ \leq& (M\delta + \frac{1}{3})\, (||u_2|| + ||v_2||)\, ||u_2 - v_2|| + \frac{1}{3}||u_1 - v_1|| \\ \leq& 2(M\delta + \frac{1}{3})\, (||u_2 - v_2|| + ||u_1 - v_1||) \\ =& 2(M\delta + \frac{1}{3})\, |||u_2 - v_2|||. \end{aligned}$$

This leads to the condition that $M\delta < 1/6$.

We find that T is a contraction on Ω if $M\delta < 1/6$ and $C = 7/6$.

Note: You can prove a better result (larger δ) by inserting $u_2^2 = 3u_2 - u_1$ in the first equation:

$$T\left(\left[\begin{array}{c} u_1 \\ u_2 \end{array}\right]\right)(x) = \left[\begin{array}{c} \int_0^\delta f(x,y)\,\big(3\,u_2(y) - u_1(y)\big)\,dy + g(x) \\ \frac{1}{3}u_1(x) + \frac{1}{3}\big(u_2(x)\big)^2 \end{array}\right].$$