Hilbert Spaces

A Hilbert space is a Banach space with an "inner product".
In other words, a Banach space where we can define a concept of angles between vectors, recall in $\mathbb{R}^n$:

$$\cancel{\cos\ \neq} \qquad (x,y) = \|x\| \, \|y\| \cos\theta$$

In this section, all linear spaces will be COMPLEX.

Def^n Let X be a complex linear space.
We say that a map $(\cdot,\cdot): X \times X \to \mathbb{C}$ is an INNER PRODUCT if

(1) $(x, \alpha y + \beta z) = \alpha(x,y) + \beta(x,z) \quad \forall \alpha,\beta \in \mathbb{C} \quad x,y,z \in X$

(2) $(x,y) = \overline{(y,x)} \quad \forall x,y \in X \quad (\Rightarrow (x,x) \in \mathbb{R})$

(3) $(x,x) \geq 0 \quad \forall x \in X$

(4) $(x,x) = 0 \iff x = 0.$

If such a map exists, we say that X is an inner product space.

Lemma $|(x,y)| \leq \sqrt{(x,x)\,(y,y)} \quad \forall x,y \in X$ $\left(\begin{array}{l} \text{Setting } \|x\| = \sqrt{(x,x)}, \text{ we have} \\ |(x,y)| \leq \|x\| \, \|y\| \end{array}\right)$

Proof: If $(x,y)=0$, then obvious, otherwise ~~Let~~ for any $\beta \in \mathbb{C}$, we have

$$0 \leq (x-\beta y, x-\beta y) = (x,x) - (x,\beta y) - (\beta y, x) + (\beta y, \beta y) =$$

$$= (x,x) - 2\operatorname{Re}\beta(x,y) + |\beta|^2 (y,y) = \qquad \text{set } \beta = \frac{\overline{(x,y)}}{|(x,y)|}\, t \ \text{ for } t \in \mathbb{R}$$

$$= (x,x) - 2t\,|(x,y)| + t^2 (y,y) = (y,y)\left[t^2 - 2t\frac{|(x,y)|}{(y,y)} + \frac{(x,x)}{(y,y)} \right] =$$

$$= (y,y)\left[\left(t - \frac{|(x,y)|}{(y,y)}\right)^2 - \frac{|(x,y)|^2}{(y,y)^2} + \frac{(x,x)}{(y,y)} \right]$$

$$\Rightarrow \frac{(x,x)}{(y,y)} - \frac{|(x,y)|^2}{(y,y)^2} \geq 0 \quad \Rightarrow \quad |(x,y)|^2 \leq (x,x)\,(y,y)$$

<u>Lemma</u> Any inner product space is a normed linear space, with the norm $\|x\| = \sqrt{(x,x)}$

<u>Proof</u> * $\|\alpha x\| = \sqrt{(\alpha x, \alpha x)} = \sqrt{\alpha\bar{\alpha}(x,x)} = |\alpha| \sqrt{(x,x)} = |\alpha| \, \|x\|$

* $\|x\| = 0 \iff (x,x) = 0 \iff x = 0$

* $\|x+y\|^2 = (x+y, x+y) = \|x\|^2 + (x,y) + (y,x) + \|y\|^2 \leq$
$$\leq \|x\|^2 + \|x\|\,\|y\| + \|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

<u>Def^n</u> A complete inner product space is called a Hilbert space.

<u>Lemma</u> If $x_n \to x$ & $y_n \to y$, then $(x_n, y_n) \to (x,y)$

<u>Proof</u> $|(x_n, y_n) - (x,y)| = |(x_n, y_n - y) + (x_n - x, y)| \leq$
$$\leq \|x_n\| \, \|y_n - y\| + \|x_n - x\| \, \|y\| \to 0 \quad \text{as } n \to \infty$$
(Recall that if $x_n \to x$, then $\sup_{n\to\infty} \|x_n\| < \infty$)

~~Rem~~ <u>Corollary</u> The inner product is a continuous map $X \times X \to \mathbb{C}$.

<u>Lemma</u> If X is an inner product space, then
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x,y \in X \qquad \text{(PAR)}$$

<u>Proof</u> $\|x+y\|^2 + \|x-y\|^2 = (x+y, x+y) + (x-y, x-y) =$
$$= \|x\|^2 + (x,y) + (y,x) + \|y\|^2 + \|x\|^2 - (x,y) - (y,x) + \|y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Lemma If $\mathfrak{X}$ is a NLS where (PAR) holds, then
$$(x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right)$$
defines an inner product on $\mathfrak{X}$ such that $\|x\| = \sqrt{(x,x)}$.

Proof: Homework.

Remark: A Banach space can be equipped with an inner product (and thus be turned into a Hilbert space) if and only if its norm satisfies (PAR).

Remark: An inner product is uniquely defined by the norm it induces. In particular, it is uniquely defined by its diagonal values.

Examples of Hilbert spaces:

(1) $\mathbb{C}^n$

(2) l^2

(3) Set $I = [0,1]$ $\mathfrak{X} =$ the set of continuous funcs on I.
Set $(f,g)_2 = \int_0^1 \overline{f(x)}\, g(x)\,dx \Rightarrow \|f\| = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2}$
$\mathfrak{X}$ is an inner product space but not a Hilbert space.
The completion of $\mathfrak{X}$ is the Hilbert space $L^2(I)$.
It consists of all functions such that $|f(x)|^2$ is Lebesgue integrable.

(4) Set $I = [0,1]$, $\mathfrak{X} =$ the set of infinitely differentiable functions on I.
Set $(f,g)_{H^n} = \sum_{j=0}^{n} \int_0^1 \overline{f^{(j)}(x)}\, g^{(j)}(x)\,dx \Rightarrow \|f\|_{H^n} = \left(\sum_{j=0}^{n} \int_0^1 |f^{(j)}(x)|^2 dx\right)^{1/2}$
$\mathfrak{X}$ is an inner product space.
The completion of $\mathfrak{X}$ is called H^n, it is a "Sobolev" space.

Orthogonality

Def^n Let $\mathfrak{X}$ be an inner product space.
If $x, y \in \mathfrak{X}$ and $(x,y)=0$, we say that x & y are ORTHOGONAL, $x \perp y$.
Let A be a subset of $\mathfrak{X}$. The ORTHOGONAL COMPLEMENT of A is
$$A^\perp = \{ y \in \mathfrak{X} : (x,y)=0 \quad \forall x \in A \}$$

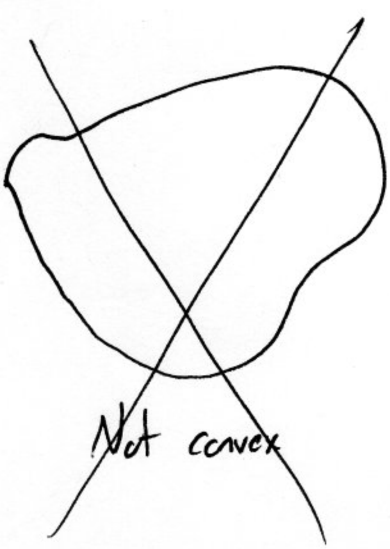
Lemma Suppose that $(x_j)_{j=1}^n \subset \mathfrak{X}$ & $x_j \perp x_k$ for $j \neq k$.
Then $\| \sum_{j=1}^n x_j \|^2 = \sum_{j=1}^n \|x_j\|^2$

Proof: ~~Trivial~~ $\| \sum_{j=1}^n x_j \|^2 = \left(\sum_{j=1}^n x_j, \sum_{k=1}^n x_k \right) = \sum_{j=1}^n \sum_{k=1}^n (x_j, x_k) = \sum_{j=1}^n \|x_j\|^2$

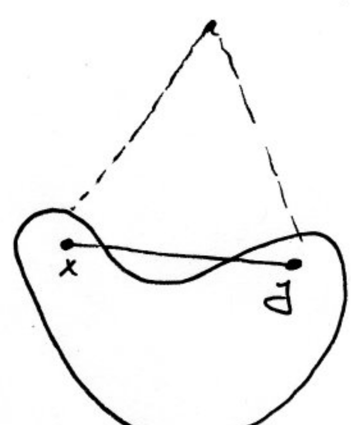
Lemma: Let A be a ~~subspace~~ subset of a Hilbert space $\mathfrak{X}$.
Then $A^\perp$ is a closed subspace of $\mathfrak{X}$.

Proof: Homework.

Def^n Let $\mathfrak{X}$ be a linear space, and M a subset of $\mathfrak{X}$.
We say that M is convex if $\forall x, y \in M$, $\alpha x + (1-\alpha) y \in M$ for $0 \le \alpha \le 1$



Not convex



Not convex

Convex

Lemma Let M be a closed convex set in a Hilbert space $\mathcal{X}$, and let x be a point in M^c.

There exists a unique element $\hat{y} \in M$ s.t. $\|x-\hat{y}\| = \inf_{y \in M} \|x-y\|$

Proof Set $d = \inf_{y\in M} \|x-y\|$. Pick $(y_n) \in M$ s.t. $\|y_n - x\| \to d$.

We will prove that (y_n) is a Cauchy seq.:

Fix $\varepsilon > 0$. Pick N s.t. $n \geq N \Rightarrow \|x-y_n\| \leq d+\varepsilon$. For $n \geq N$:

(PAR) $\Rightarrow \|y_m - y_n\|^2 = 2\|x-y_n\|^2 + 2\|x-y_m\|^2 - \|2x-y_n-y_m\|^2 \qquad (*)$

Now $\|2x - y_n - y_m\| = 2\|x - \frac{y_n+y_m}{2}\| \geq 2d$ since $\frac{y_n+y_m}{2} \in M$.

So $(*) \Rightarrow \|y_m - y_n\|^2 \leq 2(d+\varepsilon)^2 + 2(d+\varepsilon)^2 - 4d^2 = 8d\varepsilon + 4\varepsilon^2 \to 0$ as $\varepsilon \to 0$.

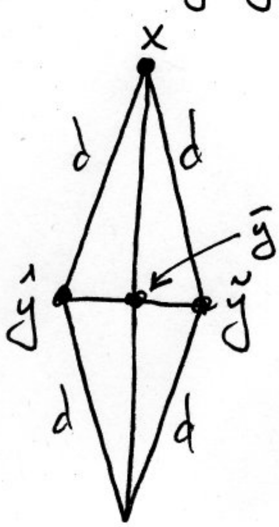
Since $\mathcal{X}$ is complete & M is closed, $\exists\ \hat{y} \in M$ s.t. $y_n \to \hat{y}$.

$\|x-\hat{y}\| = \lim_{n\to\infty} \|x-y_n\| = d$.

It only remains to prove uniqueness.

Suppose $\hat{y}$ & $\tilde{y}$ are s.t. $d = \|x-\hat{y}\| = \|x-\tilde{y}\|$

Set $a = \|\hat{y}-\tilde{y}\|$, $\bar{y} = \frac{1}{2}(\hat{y}+\tilde{y})$, and $b = \|x-\bar{y}\| \geq d$



Parallelogram law $\Rightarrow a^2 + 4b^2 = 4d^2$

But since $b \geq d$, this implies that $a = 0$

so $\hat{y} = \tilde{y}$

Def^n Let $\mathcal{X}$ be a vector space, and let A and B be subspaces of $\mathcal{X}$. We say that $\mathcal{X} = A \oplus B$ (a "direct" sum) if for any $x \in \mathcal{X}$ there exist <u>unique</u> $y \in A$, $z \in B$ such that $x = y + z$.

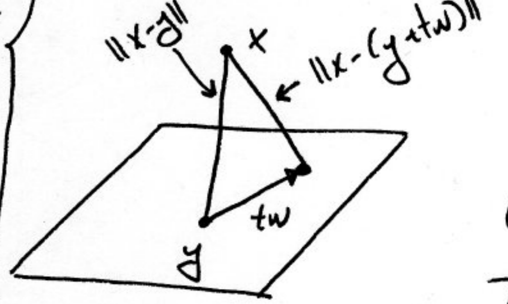
Thm Let $\mathcal{M}$ be a closed subspace of a Hilbert Space $\mathcal{X}$. Then $\cancel{\mathcal{M}}$ $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp$.

Proof Let $\mathcal{M}$ be as specified and pick $x \in \mathcal{M}^c$.

existence

Let $y \in \mathcal{M}$ be the unique element s.t. $\|x-y\| = \inf_{y' \in \mathcal{M}} \|x-y'\|$.

Set $z = x-y$. We need to prove that $z \in \mathcal{M}^\perp$. Pick any $w \in \mathcal{M}$:



Set $f(t) = \|x-(y+tw)\|^2 = \|z\|^2 - 2t\,Re(z,w) + t^2\|w\|^2$

$f(t)$ has a minimum at $t=0$. Thus

$0 = f'(0) = -2Re(z,w) \Rightarrow Re(z,w) = 0$

To prove that $Im(z,w)=0$, consider $g(t) = \|x-(y+itw)\|^2$

Uniqueness

It remains only to prove uniqueness.

Suppose that $y + z = y' + z'$ where $y, y' \in \mathcal{M}$, $z, z' \in \mathcal{M}^\perp$.

Then $y - y' = z' - z \Rightarrow \|y-y'\|^2 = (\underbrace{y-y'}_{\in \mathcal{M}}, \underbrace{z'-z}_{\in \mathcal{M}^\perp}) = 0 \Rightarrow y = y' \Rightarrow z = z'$

$\blacksquare$

In other words, given any closed subspace $\mathcal{M}$, any vector can uniquely be decomposed as $x = y + z$ where $y \in \mathcal{M}$, $z \in \mathcal{M}^\perp$.

Moreover:

$$\|x-y\| = \inf_{y' \in \mathcal{M}} \|x-y'\|$$

$$\|x-z\| = \inf_{z' \in \mathcal{M}^\perp} \|x-z'\|$$

<u>Def</u>

Def^n A projection on a linear space is a ~~lin~~ linear map P s.t. $P^2=P$.
On a Banach space, a projection must also be continuous.

Given any ~~linear~~ closed linear subspace ~~X~~ M, set $Px=y$ & $Qx=z$ where $x=y+z$ and $y\in M$, $z\in M^\perp$.

P and Q are both projections, $\|P\|=\|Q\|=1$, $PQ=QP=0$ & $P+Q=I$
$\mathrm{Ran}(P)=M$ & $\mathrm{Nul}(P)=M^\perp$.

Conversely, we have:

<u>Thm</u> Let P be a proj^n on $\underbrace{H}_{\text{a Hilbert Space}}$ with range M and nullspace N. Then

$$M\perp N \iff (Px,y)=(x,Py)\ \forall x,y\in X$$

On a Banach Space, we have the following theorems:

<u>Thm</u> Let P be a proj^n on a Banach space X, and set $M=\mathrm{Ran}(P)$, $N=\mathrm{Nul}(P)$
Then M and N are closed linear ~~sup~~ subspaces such that $X=M\oplus N$.

<u>Thm</u> Let X be a Banach space, and let M and N be closed linear subspaces such that $X=M\oplus N$. For $x\in X$, set $Px=y$ where $x=y+z$, $y\in M$, $z\in N$. Then P is a proj^n such that $M=\mathrm{Ran}(P)$, $N=\mathrm{Nul}(P)$.

<u>Remark</u>: The principal difference between Banach and Hilbert spaces in this regard is that for Hilbert spaces, we only <u>need</u> <u>one</u> subspace, namely M. Then $N=M^\perp$ always exists.
Not so for Banach spaces. There exist examples of Banach spaces X and closed linear subspaces M s.t. M is not the range of any (cont.) proj^n?

Next we prove that any Hilbert space is canonically isomorphic with itself, $\mathfrak{X} \cong \mathfrak{X}^*$.

Thm Let $\mathfrak{X}$ be a Hilbert space and assume $\varphi \in \mathfrak{X}^*$. There exists a unique $y \in \mathfrak{X}$ s.t. $\varphi(x) = (y, x)$ $\forall x \in \mathfrak{X}$.

Proof If $\varphi = 0$, then set $y = 0$, otherwise, set $\mathcal{M} = Null(\varphi)$.
Since $\mathcal{M} \neq \mathfrak{X}$ and $\mathcal{M}$ is a closed subspace, $\mathcal{M}^\perp$ is a nonempty subspace.
Pick nonzero $z \in \mathcal{M}^\perp$ ~~s.t. $\|z\| = 1$.~~
For any x, consider the vector $u = \varphi(x) z - \varphi(z) x$.
$\varphi(u) = 0$ so $u \in M$. Since $z \in \mathcal{M}^\perp$, we find that

$$0 = (z, u) = \varphi(x) \|z\|^2 - \varphi(z)(z, x) \Rightarrow \varphi(x) = \left(\frac{\varphi(z) z}{\|z\|^2}, x \right)$$

In other words, $y = \frac{\varphi(z)}{\|z\|^2} z$ works.

For uniqueness, assume $(y, x) = (y', x)$ $\forall x \Rightarrow (y - y', x) = 0$ $\forall x \overset{\text{set } x = y - y'}{\Rightarrow} y = y'$. ∎

The theorem just given is ~~the same~~ one version of the Riesz repr^n thm.

Orthogonal sets & bases

In a Hilbert space, a set $(u_\alpha)_{\alpha \in A}$ is said to be orthonormal if $\|u_\alpha\| = 1$ $\forall \alpha \in A$ & $(u_\alpha, u_\beta) = 0$ when $\alpha \neq \beta$.

Any ~~linear s~~ linearly independent seq (x_n) can be converted to an ON-seq $(u_n)_{n=1}^\infty$ such that $\mathrm{Span}(x_n)_{n=1}^N = \mathrm{Span}(u_n)_{n=1}^N$. ~~One technique is Gram-~~

Gram-Schmidt: $u_1 = \frac{1}{\|x_1\|} x_1$

$$\hat{u}_n = x_n - \sum_{j=1}^{n-1} (u_j, x_n) u_j \qquad u_n = \frac{\hat{u}_n}{\|\hat{u}_n\|}$$

Lemma (Bessel ineq.) Let $\mathfrak{X}$ be a H.S., let $(u_\alpha)_{\alpha\in A}$ be an ON-set. Then

$$\sum_{\alpha\in A} |(u_\alpha, x)|^2 \le \|x\|^2 \qquad \forall x \in \mathfrak{X}.$$

In particular, the set $\{\alpha : (u_\alpha, x) \neq 0\}$ is countable.

Proof By def^n $\displaystyle\sum_{\alpha\in A} |(u_\alpha, x)|^2 = \sup_{\substack{B\subset A \\ B \text{ is finite}}} \sum_{\alpha\in B} |(u_\alpha, x)|^2$

Set $x_\alpha = (u_\alpha, x)$, then if B is any finite subset of A:

$$0 \le \| x - \sum_{\alpha\in B} x_\alpha u_\alpha \|^2 = (x - \sum_{\alpha\in B} x_\alpha u_\alpha ,\ x - \sum_{\beta\in B} x_\beta u_\beta) =$$

$$= \|x\|^2 - \sum_{\alpha\in B} \overline{x_\alpha} \underbrace{(u_\alpha, x)}_{= x_\alpha} - \sum_{\beta\in B} x_\beta \underbrace{(x, u_\beta)}_{= \overline{x_\beta}} + \sum_{\alpha\in A} \sum_{\beta\in B} \overline{x_\alpha} x_\beta \overbrace{(u_\alpha, u_\beta)}^{= \delta_{\alpha\beta}} =$$

$$= \|x\|^2 - \sum_{\alpha\in B} |x_\alpha|^2 - \sum_{\alpha\in B} |x_\alpha|^2 + \sum_{\alpha\in B} |x_\alpha|^2 = \|x\|^2 - \sum_{\alpha\in B} |x_\alpha|^2$$

■

~~Thm~~ There are several ways of characterizing a basis for a Hilbert Space:

Thm Let $\mathfrak{X}$ be a H.S. and $\{u_\alpha\}_{\alpha\in A}$ be an ON set. TFAE:

(a) If $(u_\alpha, x) = 0$ $\forall \alpha$, then $x = 0$. (Completeness)

(b) $\|x\|^2 = \sum_{\alpha\in A} |(u_\alpha, x)|^2 \quad \forall x \in \mathfrak{X}$ (Parseval's equality)

(c) For each $x \in \mathfrak{X}$, $x = \sum_{\alpha\in A} (u_\alpha, x) u_\alpha$, where the sum has at most countably many non-zero terms and converges to x in norm regardless of the summation order.

<u>Proof</u> <u>(a)⇒(c)</u> Assume (a) holds. Fix x.

By the Bessel Lemma, at most countably many $(u_\alpha, x) \neq 0$.

Enumerate these α : $(\alpha_n)_{n=1}^{\infty}$

Set $x_n = \sum_{j=1}^{n} (u_{\alpha_j}, x) u_{\alpha_j}$. We will show that (x_n) is Cauchy.

By the Bessel ineq: $\sum_{j=1}^{\infty} |(u_{\alpha_j}, x)|^2 \leq \|x\|^2$.

For any $\varepsilon > 0$, pick N s.t. $\sum_{j=N+1}^{\infty} |(u_{\alpha_j}, x)|^2 < \varepsilon$.

Then if $m, n \geq N$ (and $m \leq n$):

$$\|x_m - x_n\|^2 = \Big\|\sum_{j=m+1}^{n} (u_{\alpha_j}, x) u_{\alpha_j}\Big\|^2 \underset{\uparrow \text{ Pyth.}}{=} \sum_{j=m+1}^{n} |(u_{\alpha_j}, x)|^2 < \varepsilon.$$

So (x_m) converges and $\sum_{j=1}^{\infty} (u_{\alpha_j}, x) u_{\alpha_j}$ exists.

Set $y = x - \sum_{j=1}^{\infty} (u_{\alpha_j}, x) u_{\alpha_j}$, then $(u_\alpha, y) = 0 \quad \forall \alpha \in A$.

By (a), this implies that $y = 0$ and so $x = \sum_{\alpha \in A} (u_\alpha, x) u_\alpha$

<u>(c)⇒(b)</u> Assume that (c) holds, ~~set~~

Given any x, set $x = \sum_{j=1}^{\infty} (u_{\alpha_j}, x) u_{\alpha_j}$.

Set $x_n = \sum_{j=1}^{n} (u_{\alpha_j}, x) u_{\alpha_j}$. Then $\|x - x_n\| \to 0$ by (c).

Moreover: $$\|x\|^2 - \sum_{j=1}^{n} |(u_{\alpha_j}, x)|^2 \underset{\uparrow \text{ By Pyth.}}{=} \|x\|^2 - \|x_n\|^2 \underset{\uparrow \text{ By Pyth.}}{=} \|x - x_n\|^2 \underset{\uparrow \text{ by (c).}}{\longrightarrow} 0$$

<u>(b)⇒(a)</u> Obvious.

<u>Def^n</u> An ON-set satisfying (a), (b), or (c), is called an ON-basis.

<u>Thm</u> Every Hilbert space has an ON-basis.

<u>Proof</u> Set $\mathcal{A}$ = the set of all ON-sets.
Then $\mathcal{A}$ is partially ordered, with <u>inclusion</u> as the order.
Each linear chain in $\mathcal{A}$ has an upper bound.
~~Thus~~, By Zorn's Lemma, there exists a maximal element, $\mathcal{M}$.
If $\mathcal{M}$ is not a basis, then $\mathcal{M}^{\perp}$ is non-empty so $\mathcal{M}$ can be extended.
Thus $\mathcal{M}$ must be a basis.

<u>Thm</u> A Hilbert space is seperable $\Leftrightarrow$ It has a countable ON-basis.
Moreover, if the space is seperable, then every ON-basis is countable.

<u>Proof</u> "$\Rightarrow$" Assume that $\mathfrak{X}$ is a seperable space.
Let $(x_n)_{n=1}^{\infty}$ be a dense set.
By discarding linearly dependent points, we construct a seq $(y_n)_{n=1}^{\infty}$ of linearly independent vectors whose span is dense.
Apply Gram-Schmidt to the set $(y_n)_{n=1}^{\infty}$ to obtain a countable basis.

"$\Leftarrow$" Assume that $\mathfrak{X}$ has a basis $(u_n)_{n=1}^{\infty}$.
Let $(q_j)_{j=1}^{\infty}$ be a dense set in $\mathbb{C}$.
Set $\Omega = \{ x = \sum_{j=1}^{J} q_j u_j \}$.
Then Ω is dense in $\mathfrak{X}$.

<u>"Moreover"</u> It remains to prove that <u>any</u> basis is countable.
Assume that $(u_n)_{n=1}^{\infty}$ is a ~~ba~~ countable basis, & $(v_\alpha)_{\alpha\in A}$ is another basis.
By the Bessel lemma, the set $A_n = \{\alpha \in A : (v_\alpha, u_n) \neq 0\}$ is countable.
Since u_n is complete, every $\alpha \in A$ belongs to at least one A_n.
Thus $A = \bigcup_{n=1}^{\infty} A_n \Rightarrow A$ is countable.

<u>Def^n</u> Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Hilbert spaces.
A bijective map $T:\mathfrak{X}\to\mathfrak{Y}$ such that

$$(Tx,Ty)_{\mathfrak{Y}} = (x,y)_{\mathfrak{X}} \quad \forall x,y\in\mathfrak{X}$$

is called a UNITARY map. When such a map exists, we say that $\mathfrak{X}$ and $\mathfrak{Y}$ are isomorphic Hilbert spaces.

<u>Example</u> Set $\mathfrak{X} = L^2((-\pi,\pi))$, $\varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, then $(\varphi_n)_{n=-\infty}^{\infty}$ is an ON-basis.
The numbers $\alpha_n = (\varphi_n, f) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-inx} f(x)\,dx$ are the Fourier coefficients.
Set $f_N = \sum_{n=-N}^{N} (\varphi_n, f)\varphi_n$, then $\|f - f_N\| \to 0$ as $N\to\infty$.
Since the inner product is continuous, we find that

$$(f,g) = \lim_{N\to\infty} (f_N, g_N) = \lim_{N\to\infty} \Big(\sum_{n=-N}^{N} (\varphi_n, f)\varphi_n, \sum_{m=-N}^{N} (\varphi_m, g)\varphi_m\Big) =$$

$$= \lim_{N\to\infty} \sum_{n=-N}^{N} \overline{(\varphi_n, f)}\,(\varphi_n, g) = \sum_{n=-\infty}^{\infty} \overline{(\varphi_n, f)}\,(\varphi_n, g) \qquad (*)$$

Define $T:\mathfrak{X}\to l^2(\mathbb{Z}) : f \mapsto ((\varphi_n, f))_{n=-\infty}^{\infty}$
Then $(*)$ shows that $(f,g)_{\mathfrak{X}} = (Tf, Tg)_{l^2(\mathbb{Z})}$
so $L^2((-\pi,\pi))$ is isomorphic to $l^2(\mathbb{Z})$.

In general, if $(u_\alpha)_{\alpha\in A}$ is any ON-basis for a Hilbert space $\mathfrak{X}$, then the map $T:\mathfrak{X}\to l^2(A) : x \mapsto ((u_\alpha, x))_{\alpha\in A}$
is a unitary isomorphism between $\mathfrak{X}$ and $l^2(A)$.

In particular, every separable Hilbert space is isomorphic to $l^2(\mathbb{N})$.