Homework set 5 — APPM5440 — Fall 2016

From the textbook: 2.7, 2.8, 2.9.

Problem 1: Set $I = (0, 1)$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of continuously differentiable functions on I. Set $\Omega = \{f_n : 1 \leq n < \infty\}$.

(a) For a given n, suppose that

$$\sup_{x \in I} |f_n'(x)| < \infty.$$

Prove that then f_n is uniformly continuous.

(b) Suppose that

$$\sup_{x \in I} \sup_{1 \leq n < \infty} |f_n'(x)| < \infty.$$

Prove that then Ω is uniformly equicontinuous.

(c) Suppose that for every $x \in I$, there exists a $\kappa > 0$ such that

$$\sup_{1 \leq n < \infty} \sup_{y \in B_\kappa(x)} |f_n'(y)| < \infty.$$

Prove that then Ω is equicontinuous.

(d) Give an example of a set Ω of functions satisfying the condition in (c) that is not uniformly equicontinuous.

(e) Suppose that for a given $x \in I$, it is the case that

$$\sup_{1 \leq n < \infty} |f_n'(x)| < \infty.$$

Prove that Ω is not necessarily equicontinuous at x.

(f) Which, if any, of the examples listed in (a) – (e) represent a bounded set Ω?

Extra problem on completeness: Fix a real number $r \in (1, \infty)$, and let X denote the set of all real-valued sequences $x = (x_1, x_2, x_3, \dots)$ such that

$$\sum_{n=1}^{\infty} |x_n|^r < \infty.$$

(a) Fix a real number $p \in [1, r)$. Define for $x \in X$, the function

$$f(x) = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

Show that $f(x) = \infty$ for some $x \in X$, and therefore f cannot be a norm on X.

(b) Fix a real number $p \in (r, \infty)$. Define for $x \in X$, the function

$$f(x) = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

Show that $f(x)$ is finite whenever $x \in X$, and that $||x|| = f(x)$ defines a norm on X. (Recall that you have already proven that $f(x+y) \leq f(x) + f(y)$ in an earlier homework.) Show that $(X, ||\cdot||)$ is not a Banach space.

(c) Repeat exercise (b) for the function

$$f(x) = \sup_{n=1,2,3,\dots} |x_n|.$$

Another problem on completeness: Set $I = [-1, 1]$ and set

$$B = \{f \in C(I) \cap C^1(I) : \sup_{x \in I} |f(x)| \leq 1 \text{ and } \sup_{x \in I} |f'(x)| \leq 1\}.$$

Consider the function $d : X \times X \to [0, \infty)$ given by

$$d(f, g) = \sup_{x \in I} |f(x) - g(x)|.$$

Show that (B, d) is a metric space that is not complete.