**Homework set 13 — APPM5440 — Fall 2016**

**Problem 1:** Let $X$ be a normed linear space, let $M$ be a closed subspace, and let $\hat{x}$ be an element not contained in $M$. Set

$$d = \text{dist}(M, \hat{x}) = \inf_{y \in M} ||y - \hat{x}||.$$

Prove that $d > 0$. Prove that there exists an element $\varphi \in X^*$ such that $\varphi(\hat{x}) = 1$, $\varphi(y) = 0$ for $y \in M$, and $||\varphi|| = 1/d$.

*Hint:* Set $Z = \text{Span}(M, \hat{x})$. Prove that any $z \in Z$ can be written $z = y + \alpha\,\hat{x}$ for a unique $\alpha \in \mathbb{R}$ and a unique vector $y \in M$. Define $\psi$ as a suitable functional on $Z$, and then extend it to $X$ using the Hahn-Banach theorem.

**Problem 2:** Let $X$ be a normed linear space with a linear subspace $M$. Prove that the weak closure of $M$ equals the closure of $M$ in the norm topology. *Hint:* Use Problem 3.

**Problem 3:** Prove that the following statements follow from the Hahn-Banach theorem:

(a) For any $x \in X$, there is a $\varphi \in X^*$ such that $||\varphi|| = 1$ and $\varphi(x) = ||x||$.

(b) For any $x \in X$, $||x|| = \sup_{||\varphi||=1} |\varphi(x)|$.

(c) If $x, y \in X$ and $x \neq y$, there is a $\varphi \in X^*$ such that $\varphi(x) \neq \varphi(y)$.

(d) For $x \in X$, define $F_x \in X^{**}$ by setting $F_x(\varphi) = \varphi(x)$.
Prove that the map $x \mapsto F_x$ is a linear isometry from $X$ to $X^{**}$.

Note that we did this in class — try to repeat the proof without looking at the notes! (We did not prove that the map $x \mapsto F_x$ is linear, you need to do this yourself.)

**Problem 4:** (Lax equivalence) Let $X$ and $Y$ be Banach spaces, let $A \in \mathcal{B}(X, Y)$ be an operator with a continuous inverse, let $f \in Y$, and consider the equation

$$A\,u = f.$$

Now suppose that we have "some mechanism" for approximating the equation to any given precision. In other words, given $\varepsilon > 0$, we can construct $A_\varepsilon$ that approximates $A$, and $f_\varepsilon$ that approximates $f$, and such that the equation

$$A_\varepsilon\,u_\varepsilon = f_\varepsilon$$

can be solved. (Typically, $A_\varepsilon$ is a finite dimensional operator, so that the approximate equation can be solved by solving a finite system of linear algebraic equations.) We say that

- The approximation is *consistent* if $A_\varepsilon \to A$ strongly.
- The approximation is *stable* if there is an $M < \infty$ such that $||A_\varepsilon^{-1}|| \leq M$ for all $\varepsilon > 0$.
- The approximation is *convergent* if $u_\varepsilon \to u$ whenever $f_\varepsilon \to f$ (in norm).

Suppose that the approximation scheme is consistent. Prove that then:

$$\text{The scheme is convergent} \quad \Leftrightarrow \quad \text{The scheme is stable}$$

*Hint:* The solution is in the text book, but please try it yourself before looking!

*Note:* In practice, variations of this result are often used in the context of approximating partial differential equations via, e.g., finite elements or finite differences. In this case, the operator is not bounded — this assumption can be done away with.