Homework set 14 — APPM5440 — Fall 2016

Note: This homework covers all of Chapter 6 — don't attempt problems until the relevant material has been covered in class.

From the textbook: 6.1, 6.2, 6.3, 6.4, 6.6. Optional: 6.12.

Problem 1: Let H be a Hilbert space, and let $(e_j)_{j=1}^n$ be an orthonormal set in H. Let x be an arbitrary vector in H. Set $M = \text{span}(e_1, \dots, e_n)$, set

$$y = \sum_{j=1}^{n} (e_j,\, x)\, e_j,$$

and set $z = x - y$. Prove that $z \in M^{\perp}$ (and consequently, that $y \perp z$). Prove that

$$||x - y|| = \inf_{y' \in M} ||x - y'||.$$

Prove that y is the *unique* minimizer (in other words, if $y' \in M \backslash \{y\}$, then $||x - y'|| > ||x - y||$). Prove these claims directly, without using the theorem about existence of a unique minimizer between a closed convex set and a point.

Problem 2: Set $I = [-1, 1]$ and consider the Hilbert space $H = L^2(I)$. Let M denote the subspace of H consisting of all even functions (in other words, functions such that $f(x) = f(-x)$ for all x). Given an $f \in H$, prove that

$$\inf_{g \in M} ||f - g|| = \left(\int_{-1}^{1} \left| \frac{f(x) - f(-x)}{2} \right|^2 dx \right)^{1/2}.$$

(Don't worry about issues relating to Lebesgue integration.)

Problem 3: Let X be a separable infinite-dimensional Hilbert space. Prove that there exists a family of closed linear subspaces $\{\Omega_t \,:\, t \in [0, 1]\}$ such that Ω_s is a strict subset of Ω_t whenever $s < t$.

Hint: It might be easier to solve the problem if you consider a particular Hilbert space, such as, e.g., $H = L^2(I)$, for $I = [0, 1]$. If you can solve the problem for this specific H, you can then invoke the theorem that all separable Hilbert spaces are unitarily equivalent.