Solution/hints for homework set 14 — APPM5440 — Fall 2016

6.1: Let M be the closed convex set in a Hilbert space H. Set $d = \inf_{x\in M} \|x\|$, and let $x_n \in M$ be such that $\|x_n\| \to d$. We will prove (x_n) is Cauchy. From the parallelogram law, we find

$$\|x_n - x_m\|^2 + \|x_n + x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2.$$

Now use that by convexity and the definition of d, we have

$$\|x_n + x_m\|^2 = 4\,\|\tfrac{1}{2}(x_n + x_m)\|^2 \geq 4d^2.$$

It follows that

$$\|x_n - x_m\|^2 \leq 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2.$$

Now use that $\|x_n\| \to d$ to prove that you can make $\|x_n - x_m\|$ arbitrarily small when $m, n \geq N$ for N large enough. Now use that H is complete to show that $x_n \to x$ for some $x \in H$.

To prove uniqueness, suppose $\|x\| = \|x'\| = d$. Then

$$\|x - x'\|^2 + \|x + x'\|^2 = 2\|x\|^2 + 2\|x'\|^2 = 4d^2.$$

Using convexity again, we find

$$\|x - x'\|^2 = 4d^2 - \|x + x'\|^2 = 4d^2 - 4\|\tfrac{1}{2}(x + x')\|^2 \leq 4d^2 - 4d^2 = 0.$$

6.2: (a) Given an function u, set $\alpha = \int_0^1 u(x)\,dx$ and $d = \inf_{n\in N} \|u - x\|$. Suppose $n \in N$. Then

$$\|u - n\| = \|(u - \alpha - n) + \alpha\| = \|g + \alpha\|, \qquad \text{where} \qquad g = u - \alpha - n.$$

Observe that $g \in N$.

<u>Case 1:</u> Suppose that n is the constant function $n(x) = \alpha$. Then $g = 0$, and we find that $\|u - n\| = \|\alpha\| = |\alpha|$. Consequently, $d \geq |\alpha|$.

<u>Case 2:</u> Suppose that n is *not* constantly α. Then $g \neq 0$. Since $\int g = 0$ and g is continuous, we know there exist points $b, c \in I$ such that $g(b) > 0$ and $g(c) < 0$. If $\alpha > 0$, then $\|g+\alpha\| \geq |g(b)+\alpha| > |\alpha|$. If $\alpha < 0$, then $\|g + \alpha\| \geq |g(c) + \alpha| > |\alpha|$.

(b) Using the results from part (a), we know that $d(u, M) \geq |\int u| = 1/2$.

To show that $d(u, M) = 1/2$, construct a minimizing sequence as follows: Pick $v_n \in M$ so that v_n is the piecewise linear continuous function that connects the nodes

$$v_n(0) = 0, \qquad v_n(1/n) = -1/2, \qquad v_n(1/2) = 0, \qquad v_n(1) = 1/2.$$

Then $\|u - v_n\| = |u(1/n) - v_n(1/n)| = 1/2 + 1/n \to 1/2$.

Finally, we prove there cannot exist an actual minimizer. Observe that we proved previously that there is a *unique* vector $v \in N$ such that $\|u - v\| = 1/2$. This is the vector $v(x) = x - 1/2$. Since this $v \notin M$, it follows that there is no actual minimizer.

6.3: First we prove that for any set A we have $A^\perp = \bar{A}^\perp$.

$\boxed{\bar{A}^\perp \subseteq A^\perp}$ This is obvious since $A \subseteq \bar{A}$. (If $x \in \bar{A}^\perp$, then x is orthogonal to every element in $\bar{A}$, which in particular implies that x is orthogonal to everything in A.)

$\boxed{A^\perp \subseteq \bar{A}^\perp}$ Suppose $x \in A^\perp$. We need to prove that for every $y \in \bar{A}$, we have $(x, y) = 0$. To this end, note that for any $y \in \bar{A}$, we can pick a sequence (y_n) such that $y_n \to y$. Then

$$(x, y) = \lim_n (x, y_n) = \{\text{Use that } y_n \in A \text{ and that } x \in A^\perp\} = \lim_n 0 = 0.$$

Next we prove that if M is a linear subspace, then $M^{\perp\perp} = \bar{M}$. Since $\bar{M}$ is a closed linear subspace, we know from a theorem in the text that

$$H = \bar{M} \oplus \bar{M}^\perp.$$

This proves that $\bar{M}^{\perp\perp} = \bar{M}$. Now invoke the result that $A^\perp = \bar{A}^\perp$ for any subset A to find that

$$\bar{M}^{\perp\perp} = \left(\bar{M}^\perp\right)^\perp = \left(M^\perp\right)^\perp = M^{\perp\perp}.$$

6.4: Proving that $H_1 \oplus H_2$ is a Hilbert space with the inner product given is a straight-forward exercise.

Set $M = \{(x_1, 0) : x_1 \in H_1\}$. We define the space N via

$$N = \{(0, x_2) : x_2 \in H_2\}.$$

We claim that $M^\perp = N$. Suppose first that $x \in N$ and $y \in M$. Then

$$(x, y) = ((0, x_2), (y_1, 0)) = (0, y_1)_{H_1} + (x_2, 0)_{H_2} = 0$$

so $x \in M^\perp$. Next, suppose that $x \notin N$. Then $x = (x_1, x_2)$ for some non-zero element x_1. Then observe that $z := (x_1, 0) \in M$ and that

$$(x, z) = ((x_1, x_2), (x_1, 0)) = (x_1, x_1)_{H_1} + (x_2, 0)_{H_2} = \|x_1\|_{H_1}^2 \neq 0,$$

so $x \notin M^\perp$.

6.6: Let us first consider the case of a *finite* orthogonal set, and let $\{x_j\}_{j=1}^n$ be such a set. Suppose that

$$c_1 x_1 + c_2 x_2 + \cdots c_n x_n = 0.$$

Then for any $j = 1, 2, \ldots, n$ we find that

$$0 = (x_j, c_1 x_1 + c_2 x_2 + \cdots c_n x_n) = c_j \|x_j\|^2$$

which proves that each c_j must equal zero.

Next consider an infinite set $\{x_j\}_{j=1}^\infty$. In this case, simply apply the finite set result to demonstrate that $\{x_j\}_{j=1}^n$ is linearly independent for any n, which shows that the full sequence in linearly independent.

Problem 1: Let H be a Hilbert space, and let $(e_j)_{j=1}^n$ be an orthonormal set in H. Let x be an arbitrary vector in H. Set $M = \text{span}(e_1, \dots, e_n)$, set

$$y = \sum_{j=1}^{n} (e_j,\, x)\, e_j,$$

and set $z = x - y$. Prove that $z \in M^{\perp}$ (and consequently, that $y \perp z$). Prove that

$$\|x - y\| = \inf_{y' \in M} \|x - y'\|.$$

Prove that y is the *unique* minimizer (in other words, if $y' \in M \backslash \{y\}$, then $\|x - y'\| > \|x - y\|$). Prove these claims directly, without using the theorem about existence of a unique minimizer between a closed convex set and a point.

Solution:

First we prove that $z \in M^{\perp}$. Set $\alpha_j = (e_j, x)$ so that

$$x = \sum_{j=1}^{n} \alpha_j e_j.$$

Let $m \in M$ so that $m = \sum_{i=1}^{n} \beta_i e_i$ for some complex numbers $\{\beta_i\}_{i=1}^{n}$. Then

$$(z, m) = (x - y, m) = (x - \sum_j \alpha_j e_j, \sum_i \beta_i e_i) = (x, \sum_i \beta_i e_i) - (\sum_j \alpha_j e_j, \sum_i \beta_i e_i) =$$
$$= \sum_i \beta_i (x, e_i) - \sum_j \overline{\alpha_j} \sum_i \beta_i (e_j, e_i) = \sum_i \beta_i \overline{\alpha_i} - \sum_j \overline{\alpha_j} \beta_j = 0.$$

Now observe that for any $m \in M$, we have

$$\|x - m\|^2 = \|(x - y) - (m - y)\|^2 = \|z - (m - y)\|^2.$$

Since $m - y \in M$, and $z \in M^{\perp}$ we find that

$$\|x - m\|^2 = \|z\|^2 + \|(m - y)\|^2.$$

This directly proves everything: $\|x - m\| \geq \|z\|$, with equality iff $m = y$.

Problem 2: Set $I = [-1, 1]$ and consider the Hilbert space $H = L^2(I)$. Let M denote the subspace of H consisting of all even functions (in other words, functions such that $f(x) = f(-x)$ for all x). Given an $f \in H$, prove that

$$\inf_{g \in M} \|f - g\| = \left(\int_{-1}^{1} \left| \frac{f(x) - f(-x)}{2} \right|^2 \, dx \right)^{1/2}.$$

(Don't worry about issues relating to Lebesgue integration.)

Solution:

Set

$$d = \inf_{g \in M} \|f - g\|.$$

Then define two maps P and Q on H via

$$[Pf](x) = \frac{1}{2}\big(f(x) + f(-x)\big),$$
$$[Qf](x) = \frac{1}{2}\big(f(x) - f(-x)\big).$$

Then for any $f \in H$, we have

$$f = Pf + Qf.$$

Next, observe that for any $g \in M$ we have

$$\|f - g\|^2 = \|(f - Pf) - (g - Pf)\|^2 = \|Qf - (g - Pf)\|^2.$$

Now observe that since Qf is odd and $g - Pf$ is even, we have $Qf \perp (g - Pf)$. Consequently

$$\|f - g\|^2 = \|Qf\|^2 + \|(g - Pf)\|^2.$$

This relation proves directly that $d = \|Qf\|$ and that the minimum is attained iff $g = Pf$.

Problem 3: Let X be a separable infinite-dimensional Hilbert space. Prove that there exists a family of closed linear subspaces $\{\Omega_t : t \in [0,1]\}$ such that Ω_s is a strict subset of Ω_t whenever $s < t$.

Hint: It might be easier to solve the problem if you consider a particular Hilbert space, such as, e.g., $H = L^2(I)$, for $I = [0,1]$. If you can solve the problem for this specific H, you can then invoke the theorem that all separable Hilbert spaces are unitarily equivalent.

Solution:

In the space H suggested in the hint, the problem is easy. Set, e.g.,

$$\Psi_t = \{f \in L^2(I) : f(x) = 0 \text{ when } x > t\}.$$

Then the family $\{\Psi_t\}_{t\in[0,1]}$ satisfies the criteria.

For a general Hilbert space X, let $U : X \to H$ be a unitary map. Then set

$$\Omega_t = U^{-1}\,\Psi_t\,U.$$