**APPM5440 — Applied Analysis: Section exam 1 — Solutions**
10:00 – 10:50, Sep. 23, 2016. Closed books.

*Each problem has a max of 8 points, for a total maximum score of 40 points.*

**Problem 1:** Consider the set $X = \mathbb{R}^3$. Let $p$ be a real number such that $0 < p < \infty$.

(a) For which values of $p$ in the interval $(0, \infty)$ is the following function a metric on $X = \mathbb{R}^3$:
$$d(x, y) = \big(|x_1 - y_1|^p + |x_2 - y_2|^p + |x_3 - y_3|^p\big)^{1/p}.$$

(b) For which values of $p$ in the interval $(0, \infty)$ is the following function a metric on $X = \mathbb{R}^3$:
$$d(x, y) = \big(|x_1 - y_1|^p + |x_2 - y_2|^p\big)^{1/p} + |x_3 - y_3|.$$

(c) For which values of $p$ in the interval $(0, \infty)$ is the following function a metric on $X = \mathbb{R}^3$:
$$d(x, y) = |x_1 - y_1|^p + |x_2 - y_2| + |x_3 - y_3|.$$

No motivation is necessary, just write down your answer to each part. Observe carefully that the question is about *metrics,* not *norms.*

*Solution:*

(a) $\boxed{p \in [1, \infty)}$ This is just the standard $\ell^p$ norm on $\mathbb{R}^3$. When $p < 1$, the triangle inequality does not hold.

(b) $\boxed{p \in [1, \infty)}$ This is a mixture of the $\ell^1$ and $\ell^p$ norms. If $p < 1$, then the triangle inequality does not hold in the $x_1$-$x_2$-plane. If $p \geq 1$, then observe that if $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are two NLS, then $X_1 \times X_2$ is a NLS with the norm $\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2$.

(c) $\boxed{p \in (0, 1]}$ See the proof of Exercise 1.5. We showed if $(X, \|\cdot\|)$ is a NLS, and if $f : [0, \infty) \to [0, \infty)$ is a function such that $f(0) = 0$, $f' \geq 0$, and $f'$ is non-increasing, then $d(x, y) = f(\|x - y\|)$ is a metric on $X$. With $f(t) = t^p$, we see that the conditions hold if $p \in (0, 1]$. If $p > 1$, then the triangle inequality is violated on the $x_1$-axis.

**Problem 2:** Let $(X, d)$ be a metric space, and let $\Omega$ be a subset of $X$.

(a) Define what it means for $\Omega$ to be *totally bounded.*
(b) Suppose that $X$ itself is totally bounded. Does $X$ necessarily have a countable dense subset? If you answer yes, then prove this. If you answer no, then provide a counter example.

*Solution:*

For (a) see the textbook. The answer to (b) is yes, and the proof is a subset of the proof that any compact set has a dense countable subset. Observe that we do *not* need $X$ to be complete for this proof to work.

*Max of 2p for (a), and 6p for (b).*

**Problem 3:** In this problem, let $(X, d)$ denote a metric space.

(a) Let $\Omega$ be a subset of $X$. State the definition of the *closure* of $\Omega$.

(b) Consider the set of rational numbers $X = \mathbb{Q}$ equipped with the standard metric (the absolute value function). Set $\Omega = \{x \in X : x^2 < 2\}$. What is the closure of $\Omega$?

(c) State the definition of a *completion* of $(X, d)$.

(d) Consider the set $X$ of *positive* rational numbers. What is the completion of $X$? (The completion is not unique, of course, but there is one very natural candidate.)

*Solution:*

For (b), the answer is that $\overline{\Omega} = \Omega$. Recall that in one of the homeworks, we proved that in $X$, the set $\Omega$ is closed, so it is its own closure. Recall that when we talk about closure of a set (as opposed to "completion") then we only work with points already in the set. So in this particular example, we cannot add non-rational numbers to the closure.

For (d), one natural completion is the interval $I = [0, \infty)$ of real numbers. Observe that $\infty$ should *not* be included in the completion since there is no Cauchy sequence in $X$ that converges to $\infty$. (Every Cauchy sequence is bounded.)

*Max of 2p for each sub-question.*

**Problem 4:** Set $X = \ell^2$. In other words, an element $x \in X$ if it is a sequence of real numbers $x = (x_1, x_2, x_3, \dots)$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. The norm on $X$ is $\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$. Let $B$ denote the unit ball, so that $B = \{x \in X : \|x\| \leq 1\}$. Prove that $B$ is not a compact set.

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*Solution:*

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Let $\{e^{(n)}\}_{n=1}^{\infty}$ denote the canonical basis vectors, so that

$$\begin{aligned} e^{(1)} =& (1, 0, 0, 0, \dots), \\ e^{(2)} =& (0, 1, 0, 0, \dots), \\ e^{(3)} =& (0, 0, 1, 0, \dots), \end{aligned}$$

Then you can easily verify that $e^{(n)} \in B$ for every $n$, and that if $m \neq n$, then $\|e^{(n)} - e^{(m)}\| = \sqrt{2}$, which shows that $(e^{(n)})$ cannot have a convergent subsequence. Alternative, you could use these observations to prove that if $\varepsilon < 1/\sqrt{2}$, then there cannot exist a finite $\varepsilon$-cover of $B$ since at most one $e^{(n)}$ can be inside any such $\varepsilon$-ball.

**Problem 5:** Set $I = [0,1]$ and let $X$ denote the set of real-valued piecewise continuous functions $f$ on $I$ such that

$$\int_0^1 |f(x)|^2\,dx < \infty.$$

(Since $f$ is piecewise continuous, this is a plain Riemann integral.) Define the function $n$ on $X$ via

$$n(f) = \int_0^1 |f(x)|\,dx.$$

(a) Prove that the function $n$ is a seminorm on $X$.
(b) Construct a sequence of functions $(f_n)_{n=1}^{\infty}$ in $X$ that is Cauchy with respect to $n$, and that converges pointwise to a function on $I$ that does not belong to $X$.

*Solution:*

For (b), you could use, for instance, the functions

$$f_n(x) = \begin{cases} 0, & x \in [0, 1/n), \\ x^{-1/2}, & x \in [1/n, 1]. \end{cases}$$

We have $\int_0^1 |f_n|^2 = \int_{1/n}^1 x^{-1}\,dx = \log(n) < \infty$ so $f_n \in X$. Moreover, for $m \leq n$, we have

$$\|f_n - f_m\| = \int_0^1 |f_n(x) - f_m(x)|\,dx = \int_{1/n}^{1/m} x^{-1/2}\,dx = [2\,x^{1/2}]_{1/n}^{1/m} = 2\,m^{-1/2} - 2\,n^{-1/2}.$$

We see that if $m, n \geq N$, then $\|f_n - f_m\| \leq 2\,N^{-1/2}$ so $(f_n)$ is indeed Cauchy. The pointwise limit of $(f_n)$ is the function

$$f(x) = \begin{cases} 0, & x = 0, \\ x^{-1/2}, & x \in (0, 1]. \end{cases}$$

But

$$\int_0^1 |f(x)|^2\,dx = \int_0^1 x^{-1}\,dx = \infty,$$

so $f \notin X$.

*Max of 2p for (a), and 6p for (b).*