APPM5440 — Applied Analysis: Section exam 3

17:00 – 18:15, Nov. 30, 2012. Closed books.

WRITE YOUR NAME: ____

Fill out your answers to problems 1 and 2 directly on the problem sheet. No motivations required.

Write your answers to problems 3 and 4, with motivations, either on the exam, or on separate sheets.

Problem 1: (10p) No motivation required — please just write the answers. 2p per problem.

(a) Let X be any set, and let $\mathcal{T}$ be a collection of subsets of X. Write the conditions that $\mathcal{T}$ must satisfy in order to be a *topology* on X.

(b) For which values of p is the Banach space ℓ^p reflexive? Answer: ______________________________

(c) Let X and Y be normed linear spaces. Mark the true statements:

	Check if true:
For $\mathcal{B}(X,Y)$ to be complete, it is sufficient for X to be complete.	
For $\mathcal{B}(X,Y)$ to be complete, it is sufficient for Y to be complete.	
For $\mathcal{B}(X,Y)$ to be complete, both X and Y must be complete.	

(d) Set $I = [-\pi, \pi]$ and $X = C(I)$. Consider the operator $T \in \mathcal{B}(X)$ defined by

$$[Tf](x) = \int_{-\pi}^{\pi} \sin(x-y)\, f(y)\, dy, \qquad x \in I.$$

Determine the range of T. Answer: $\text{ran}(T) =$ ______________________________

(e) Let c be a real number, let $X = \ell^2$, and define the operator $T \in \mathcal{B}(X)$ via

$$T(x_1,\, x_2,\, x_3,\, x_4,\, \dots) = \big(\big(c + \tfrac{1}{1}\big)\, x_1,\ \big(c + \tfrac{1}{2}\big)\, x_2,\ \big(c + \tfrac{1}{3}\big)\, x_3,\ \big(c + \tfrac{1}{4}\big)\, x_4,\ \dots\big).$$

For which values of c is the range of T closed?

Answer: ______________________________

Problem 2: (10p) Set $I = [0, 2]$ and let X denote the space of continuous functions on I. Define a functional on X via $\varphi(f) = \displaystyle\int_0^2 x^2\, f(x)\, dx$.

(a) (5p) Equip X with the norm $\|f\| = \sup_{x \in I} |f(x)|$. Compute $\|\varphi\|_{X^*}$. Answer: $\|\varphi\|_{X^*} =$ ______________

(b) (5p) Equip X with the norm $\|f\| = \displaystyle\int_0^2 |f(x)|\, dx$. Compute $\|\varphi\|_{X^*}$. Answer: $\|\varphi\|_{X^*} =$ ______________

Problem 3: (10p) Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f((x_1, x_2)) = x_1$. We use the standard Euclidean norm on both $\mathbb{R}^2$ and on $\mathbb{R}$.

(a) (5p) Prove that f is open.
(b) (5p) Prove that f does not necessarily map closed sets to closed sets.

Problem 4: (10 points) Let X denote a Banach space.

(a) (3p) Let $\{T_n\}_{n=1}^{\infty}$ be a sequence $\mathcal{B}(X)$. Define the following concepts:
  (i) $\{T_n\}$ converges *in norm.*
  (ii) $\{T_n\}$ converges *strongly.*
  (iii) $\{T_n\}$ converges *weakly.*

(b) (5p) Let $X = \ell^1$ with the usual norm. Consider the sequence of operators $\{T_n\}_{n=1}^{\infty}$ defined by

$$T_n(x_1,\, x_2,\, x_3,\, \dots) = (x_1,\, x_2,\, x_3,\, \dots,\, x_{n-1},\, x_n,\, 0,\, 0,\, 0,\, \dots).$$

Does $\{T_n\}$ converge in any of the three modes? Please motivate your answer.

(c) (2p) Let $\{T_n\}_{n=1}^{\infty}$ denote the same operators as in (b), but now acting on $X = \ell^\infty$. Does $\{T_n\}_{n=1}^{\infty}$ converge strongly?