Hand in solutions to at least three of the following four problems.

**Question 2.1:** Consider three different ways of computing the DFT:

(a) Apply it directly via the formula

\[
\hat{x}(n) = [F_N x](n) = \sum_{m=0}^{N-1} e^{-i 2\pi m n/N} x(m), \quad n = 0, 1, 2, \ldots, N - 1.
\]

(It is strongly recommended to vectorize the matrix-vector product.)

(b) Build your own implementation of the basic FFT algorithm described in the course notes. (This is the Cooley-Tukey algorithm.)

(c) Apply the built-in Matlab command FFT. (Last time I checked, Matlab used the FFTW package, but this may have changed.)

Measure the asymptotic timing for each of the methods, and comment on what you observe.

**Question 2.2:** Solve (analytically) the Laplace equation with Dirichlet boundary data on a domain exterior to a unit disc. In other words, let \( \Omega \) and \( \Gamma \) be defined by

\[
\begin{align*}
\Omega &= \{ x \in \mathbb{R}^2 : |x| > 1 \}, \\
\Gamma &= \{ x \in \mathbb{R}^2 : |x| = 1 \},
\end{align*}
\]

and consider the boundary value problem

\[
\begin{cases}
-\Delta u(x) = 0, & x \in \Omega, \\
u(x) = g(x), & x \in \Gamma.
\end{cases}
\]

You may assume that \( g \in L^1 \cap L^2 \), that

\[
\int_0^{2\pi} g(\cos \theta, \sin \theta) = 0,
\]

and then require the solution \( u \) to decay at infinity.
Question 2.3: In this problem, we will numerically solve the equation
\[
\begin{align*}
-u''(x) &= f(x), \quad x \in (0, \pi), \\
u(0) &= 0, \\
u(\pi) &= 0,
\end{align*}
\]
for a few different right hand sides. For each given \( f \), compute the solution using either the basis functions \( \{\sqrt{2/\pi} \sin(nx)\}_{n=1}^{\infty} \), or the basis functions \( \{\sqrt{1/\pi} e^{2inx}\}_{n \in \mathbb{Z}} \) (with appropriate corrections for the boundary conditions). Consider the following choices of \( f \):

(a) \( f(x) = e^{(\sin x)^2} \).

(b) \( f(x) = \sqrt{x} \).

(c) \( f(x) = \cos(100x) \).

(d) \( f(x) = \cos(100.1x) \).

(e) \( f(x) = |x - 1|^{-0.25} \).

(f) \( f(x) = \begin{cases} 
0 & 0 < x \leq 1 \\
1 & 1 < x \leq 2 \\
2 & 1 < x < \pi
\end{cases} \)

Your solution should contain the following: (i) A plot of the solution \( u \). (ii) Plots of \( \hat{f}_n \) and \( \hat{u}_n \) versus \( n \) for each of the functions and a discussion of the rates of decay.

Question 2.4: Construct a program that uses Fourier methods to solve the heat equation
\[
\begin{align*}
\frac{d^2u}{dx^2} - \frac{1}{c^2} \frac{du}{dt}, & \quad x \in (0, \pi), t > 0, \\
u(0, t) &= 0, \quad t > 0, \\
u(\pi, t) &= 0, \quad t > 0, \\
u(x, 0) &= f(x), \quad 0 < x < \pi \\
u_t(x, 0) &= 0,
\end{align*}
\]
where \( f(x) = e^{(\sin x)^2} \) or \( f(x) = |x - 1|^{-0.25} \). For both choices of \( f \), produce plots of the solution at a few different times \( t \) (your choice!). In addition, produce a plot of the function
\[
U(t) = \int_0^\pi |u(x, t)|^2 \, dx.
\]
Repeat the exercise, but now consider the wave equation
\[
\begin{align*}
\frac{d^2v}{dx^2} - \frac{1}{c^2} \frac{dv}{dt}, & \quad x \in (0, \pi), t > 0, \\
v(0, t) &= 0, \quad t > 0, \\
v(\pi, t) &= 0, \quad t > 0, \\
v(x, 0) &= f(x), \quad 0 < x < \pi \\
v_t(x, 0) &= 0,
\end{align*}
\]
Produce a plot of
\[
V(t) = \int_0^\pi |v(x, t)|^2 \, dx.
\]

Hint: For your own amusement, you may want to create animations of the solutions using the Matlab movie command.