The problems in this homework rely on the geometry shown below:

\[
\begin{array}{c}
\text{a} \bullet \\
\Psi
\end{array}
\begin{array}{c}
\text{c} \bullet \\
\text{d} \bullet \\
\text{e} \bullet \\
\Omega
\end{array}
\begin{array}{c}
\text{b} \bullet \\
\Gamma
\end{array}
\]

The contour \( \Gamma \) is defined via
\[
\Gamma = \{ x = (G_1(t), G_2(t)) : t \in [0, 2\pi] \}.
\]

where
\[
G_1(t) = 1.5 \cos(t) + 0.1 \cos(6t) + 0.1 \cos(4t),
\]
\[
G_2(t) = \sin(t) + 0.1 \sin(6t) - 0.1 \sin(4t).
\]

The coordinates of the points are
\[
a = (-2, 1) \quad b = (2, 1) \quad c = (1.7, 0) \quad d = (0.3, 0.5) \quad e = (-0.1, 0.2).
\]

The domain \textit{interior} to \( \Gamma \) is \( \Omega \), and the domain \textit{exterior} to \( \Gamma \) is \( \Psi \).

**Problem 5.1:** Consider the \textit{exterior} Neumann problem

\[
\begin{align*}
-\Delta u(x) &= 0, \quad x \in \Psi, \\
u_n(x) &= r(x), \quad x \in \Gamma,
\end{align*}
\]

where
\[
f(x_1, x_2) = x_1 e^{\sin(10x_2)},
\]

and where \( r \) is defined to equal \( f \), but shifted so that \( \int_{\Gamma} r = 0 \):
\[
r(x) = f(x) - \frac{1}{|\Gamma|} \int_{\Gamma} f(x) dl(x).
\]

Let \( u \) have the representation
\[
u(x) = [S\sigma](x) = \int_{\Gamma} \frac{1}{2\pi} \log \frac{1}{|x - x'|} \sigma(x') dl(x').
\]

Your task is to form an equation for \( \sigma \), discretize this equation, solve the equation, and then to evaluate the function \( u \). (You will find the relevant formulas in the course notes!)

Your answer should include a print-out of your Matlab code, and an accurate estimate of
\[
u(a) - u(b).
\]

(Observe that \( u(a) \) and \( u(b) \) are not uniquely determined by the Neumann problem, but their \textit{difference} is!)
Problem 5.2: Repeat Problem 5.1, but now solve the corresponding interior problem

\[
\begin{cases}
-\Delta u(x) = 0, & x \in \Omega, \\
u_n(x) = r(x), & x \in \Gamma,
\end{cases}
\]

where \(\Omega\) is the domain interior to \(\Gamma\).

First look for a solution of the form

\[
u(x) = [S \sigma](x) = \int_{\Gamma} \frac{1}{2\pi} \log \frac{1}{|x - x'|} \sigma(x') \, dl(x').
\]

This will result in a linear system

\[ A \sigma = r \]

where \(A\) is an \(N \times N\) matrix of rank \(N - 1\). Verify that \(r \in \text{Col}(A)\) (the column space, or range, of \(A\)), and then construct a solution via

\[
\sigma = A^\dagger r,
\]

where \(A^\dagger\) is the Moore-Penrose pseudo-inverse

\[
A^\dagger = V(:,1:(N-1)) \Sigma(1:(N-1),1:(N-1))^{-1} U(:,1:(N-1))^*.
\]

is the SVD of \(A\).

Next look for a solution of the form

\[
u(x) = [S \sigma](x) = \int_{\Gamma} \frac{1}{2\pi} \log \frac{1}{|x - x'|} \sigma(x') \, dl(x') + \frac{1}{2\pi} \left( \log \frac{1}{|x|} \right) \int_{\Gamma} \sigma(x') \, dl(x').
\]

(For a motivation of this choice, see course notes.)

In your answer, simply specify the value of

\[ u(d) - u(e). \]