

The outgoing expansion

Let τ be a box (green).

Let \mathbf{c}_τ be the center of τ (black).

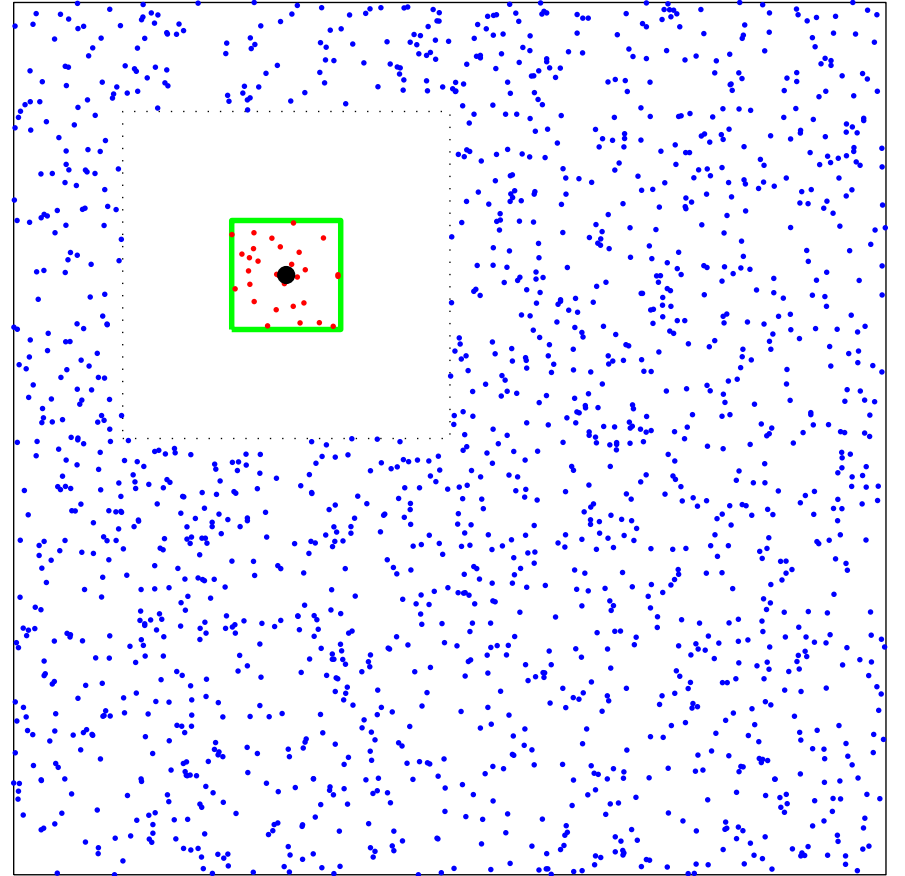
Let \mathbf{y}_j be source locations in τ (red).

Let q_j be the strength of source j .

Let \mathbf{x}_i be targets well separated from τ (blue).

Let u denote the potential

$$u(\mathbf{x}_i) = \sum_j q_j \log(\mathbf{x}_i - \mathbf{y}_j).$$



The *outgoing expansion* of τ is a vector $\hat{\mathbf{q}} = [\hat{q}_p]_{p=0}^P$ of complex numbers such that

$$(1) \quad u(\mathbf{x}) \approx \hat{q}_0 \log |\mathbf{x} - \mathbf{c}_\tau| + \sum_{p=1}^P \hat{q}_p \frac{1}{(\mathbf{x} - \mathbf{c}_\tau)^p}, \quad \mathbf{x} \in \Omega_\tau^{\text{far}}.$$

The outgoing expansion is a compact representation of the sources inside τ (it encodes both the source locations and the magnitudes).

The incoming expansion

Let τ be a box (green).

Let \mathbf{c}_τ be the center of τ (black).

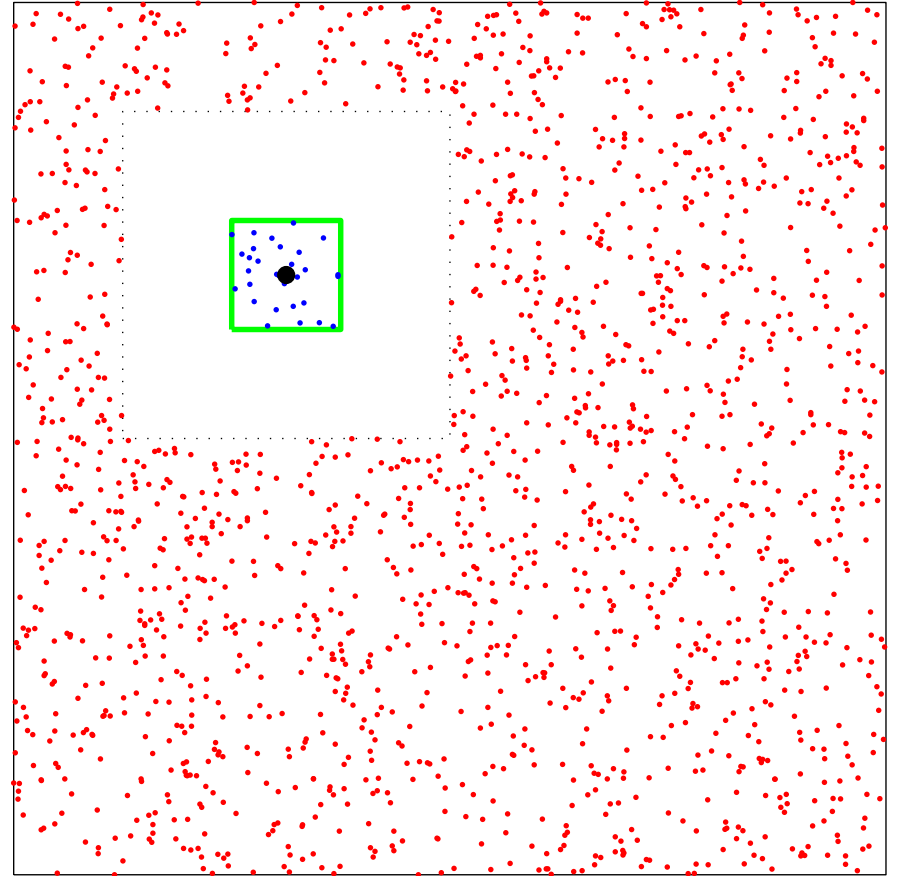
Let \mathbf{y}_j be sources well-separated from τ (red).

Let q_j be strength of source j .

Let \mathbf{x}_i be targets inside τ (blue).

Let u denote the potential

$$u(\mathbf{x}_i) = \sum_j q_j \log(\mathbf{x}_i - \mathbf{y}_j).$$



The *incoming expansion* of τ is a vector $\hat{\mathbf{u}} = [\hat{u}_p]_{p=0}^P$ of complex numbers such that

$$(2) \quad u(\mathbf{x}) \approx \sum_{p=0}^P \hat{u}_p (\mathbf{x} - \mathbf{c}_\tau)^p, \quad \mathbf{x} \in \Omega_\tau.$$

The incoming expansion is a compact representation of the sources well-separated from τ (it encodes both the source locations and the magnitudes).

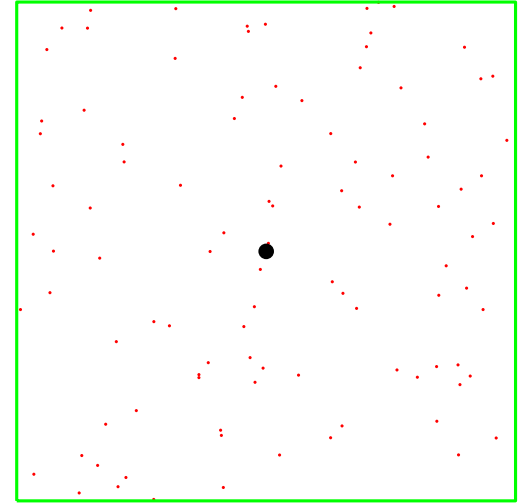
The *outgoing-from-sources* translation operator $\mathsf{T}_\tau^{(\text{ofs})}$

Let τ be a box (green).

Let \mathbf{c}_τ be the center of τ (black).

Let $\{\mathbf{y}_j\}_j^{N_\tau}$ be source locations in τ (red).

Let q_j be strength of source j .



The operator $\mathsf{T}_\tau^{(\text{ofs})}$ constructs the outgoing expansion directly from the vector of charges.

$$\begin{array}{ccc} \hat{\mathbf{q}}_\tau & = & \mathsf{T}_\tau^{(\text{ofs})} \mathbf{q} \\ (P+1) \times 1 & & (P+1) \times N_\tau \quad N_\tau \times 1 \end{array}$$

$$\begin{array}{ll} \mathsf{T}_{\tau,0,j}^{(\text{ofs})} = 1 & 1 \leq j \leq N_\tau \\ \mathsf{T}_{\tau,p,j}^{(\text{ofs})} = -\frac{1}{p}(\mathbf{y}_j - \mathbf{c}_\tau)^p & 1 \leq p \leq P \quad 1 \leq j \leq N_\tau. \end{array}$$

The *outgoing-from-outgoing* translation operator $\mathsf{T}_{\tau,\sigma}^{(\text{of})}$

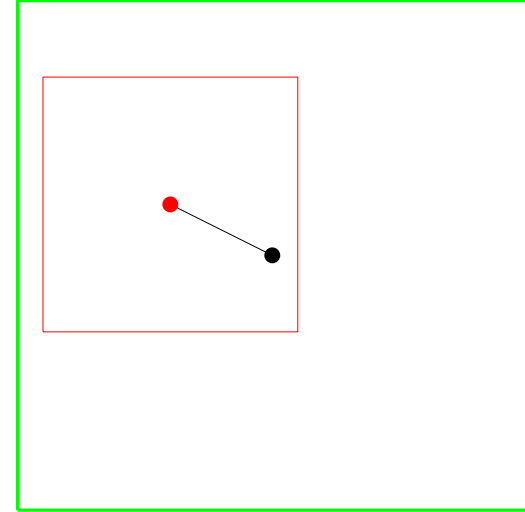
Let τ be a box (green).

Let \mathbf{c}_τ be the center of τ (black).

Let σ denote a box contained in τ .

Let \mathbf{c}_σ denote the center of σ (red).

Let $\hat{\mathbf{q}}_\sigma$ be outgoing expansion of σ .



$\mathsf{T}_{\tau,\sigma}^{(\text{of})}$ constructs the outgoing expansion of τ from the outgoing expansion of σ

$$\begin{array}{ccc} \hat{\mathbf{q}}_\tau & = & \mathsf{T}_{\tau,\sigma}^{(\text{of})} \hat{\mathbf{q}}_\sigma \\ (P+1) \times 1 & & (P+1) \times (P+1) \quad (P+1) \times 1 \end{array}$$

With $\mathbf{d} = \mathbf{c}_\sigma - \mathbf{c}_\tau$, $\mathsf{T}_{\tau,\sigma}^{(\text{of})}$ is a lower tridiagonal matrix with entries

$$\mathsf{T}_{\tau,\sigma,0,0}^{(\text{of})} = 1$$

$$\mathsf{T}_{\tau,\sigma,p,0}^{(\text{of})} = -\frac{1}{p} \mathbf{d} \quad 1 \leq p \leq P$$

$$\mathsf{T}_{\tau,\sigma,p,q}^{(\text{of})} = \binom{p}{q} \mathbf{d}^{p-q} \quad 1 \leq q \leq p \leq P.$$

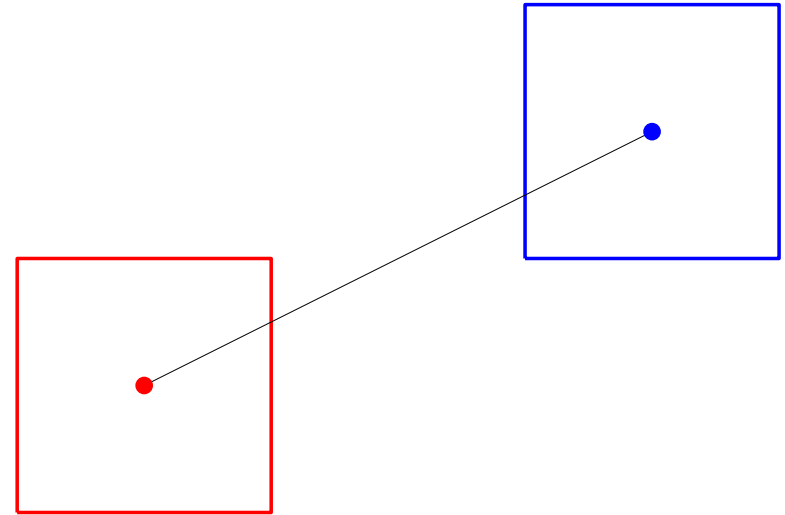
The *incoming-from-outgoing* translation operator $\mathsf{T}_{\tau,\sigma}^{(\text{ifo})}$

Let σ be a source box (red) with center \mathbf{c}_σ .

Let τ be a target box (blue) with center \mathbf{c}_τ .

Let $\hat{\mathbf{q}}_\sigma$ be the outgoing expansion of σ .

Let $\hat{\mathbf{u}}_\tau$ represent the potential in τ caused by sources in σ .



$\mathsf{T}_{\tau,\sigma}^{(\text{ifo})}$ constructs the incoming expansion of τ from the outgoing expansions of σ :

$$\begin{array}{ccccc} \hat{\mathbf{u}}_\tau & = & \mathsf{T}_{\tau,\sigma}^{(\text{ifo})} & \hat{\mathbf{q}}_\sigma & \\ (P+1) \times 1 & & (P+1) \times (P+1) & (P+1) \times 1 & \end{array}$$

With $\mathbf{d} = \mathbf{c}_\sigma - \mathbf{c}_\tau$, $\mathsf{T}_{\tau,\sigma}^{(\text{ifo})}$ is a matrix with entries

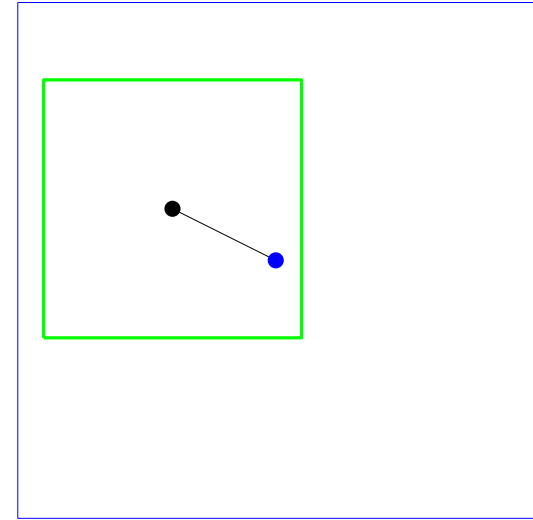
$$\mathsf{T}_{\tau,\sigma,p,q}^{(\text{ifo})} = ?$$

The *incoming-from-incoming* translation operator $\mathbb{T}_{\tau,\sigma}^{(\text{ifi})}$

Let τ be a box (green) with center \mathbf{c}_τ (black).

Let σ be a box (blue) containing τ with center \mathbf{c}_σ .

Let $\hat{\mathbf{u}}_\sigma$ be an incoming expansion for σ .



$\mathbb{T}_{\tau,\sigma}^{(\text{ifi})}$ constructs the incoming expansion of τ from the incoming expansion of σ

$$\begin{array}{ccccc} \hat{\mathbf{u}}_\tau & = & \mathbb{T}_{\tau,\sigma}^{(\text{ifi})} & & \hat{\mathbf{u}}_\sigma \\ (P+1) \times 1 & & (P+1) \times (P+1) & & (P+1) \times 1 \end{array}$$

With $\mathbf{d} = \mathbf{c}_\sigma - \mathbf{c}_\tau$, $\mathbb{T}_{\tau,\sigma}^{(\text{ifi})}$ is a matrix with entries

$$\mathbb{T}_{\tau,\sigma,p,q}^{(\text{ifi})} = ?$$

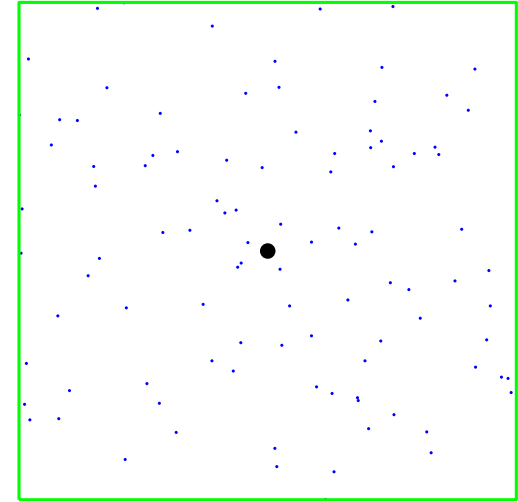
The *targets-from-incoming* translation operator $\mathsf{T}_\tau^{(\text{tfi})}$

Let τ be a box (green).

Let \mathbf{c}_τ be the center of τ (black).

Let $\{\mathbf{x}_i\}_i^{N_\tau}$ be target locations in τ (blue).

Let $\hat{\mathbf{u}}_\tau$ be the incoming expansion of τ .



$\mathsf{T}_\tau^{(\text{tfi})}$ constructs the potentials in τ from the incoming expansion

$$\begin{array}{ccccc} \mathbf{u}_\tau & = & \mathsf{T}_\tau^{(\text{tfi})} & & \hat{\mathbf{u}}_\tau \\ N_\tau \times 1 & & N_\tau \times (P + 1) & (P + 1) \times 1 & \end{array}$$

$$\mathsf{T}_{\tau,i,p}^{(\text{tfi})} = (\mathbf{x}_i - \mathbf{c}_\tau)^p \quad 1 \leq i \leq N_\tau \quad 0 \leq p \leq P.$$

How do you compute the expansions of a box?

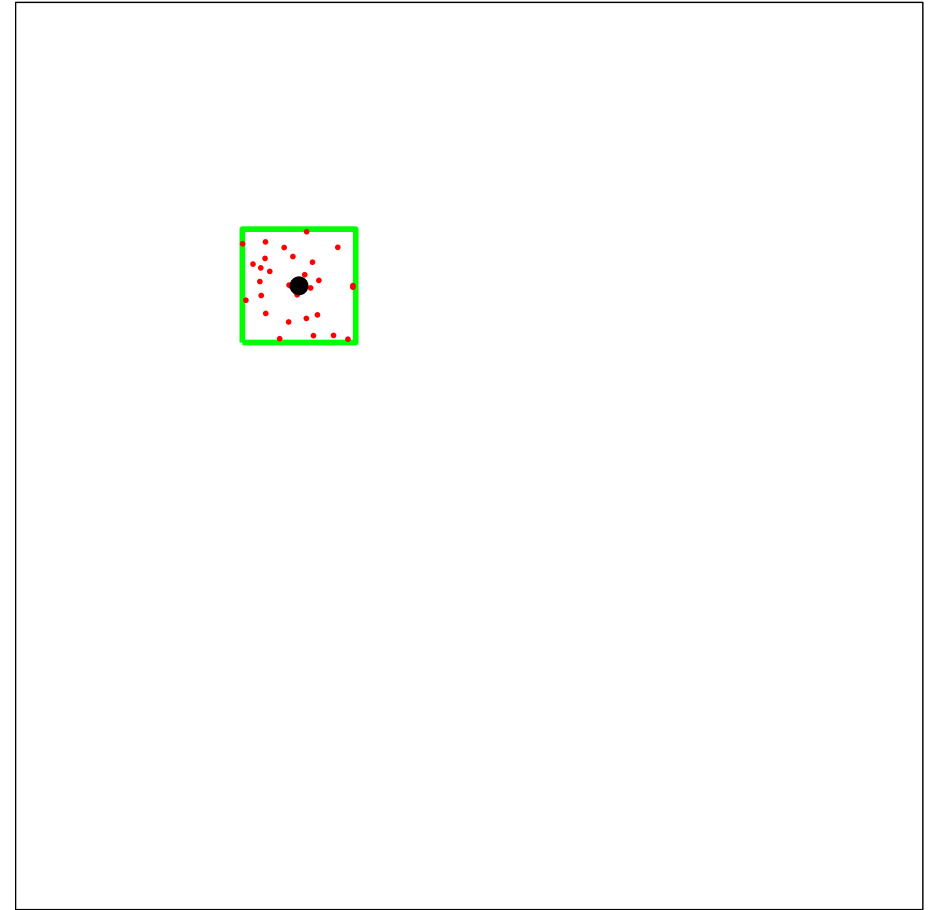
Computing the outgoing expansion of a leaf

Let τ be a box (green).

Let \mathbf{c}_τ be the center of τ (black).

Let $\{\mathbf{y}_j\}_j^{N_\tau}$ be source locations in τ (red).

Let q_j be strength of source j .



There is an analytic formula:

$$\hat{q}_0 = \sum_{j=1}^{N_\tau} q_j \quad \hat{q}_p = -\frac{1}{p} \sum_{j=1}^{N_\tau} q_j (\mathbf{y}_j - \mathbf{c}_\tau)^p, \quad p = 1, 2, \dots, P.$$

We write the formula compactly as

$$\hat{\mathbf{q}}_\tau = \mathbb{T}_\tau^{(\text{ofs})} \mathbf{q}_\tau.$$

Computing the outgoing expansion of a parent

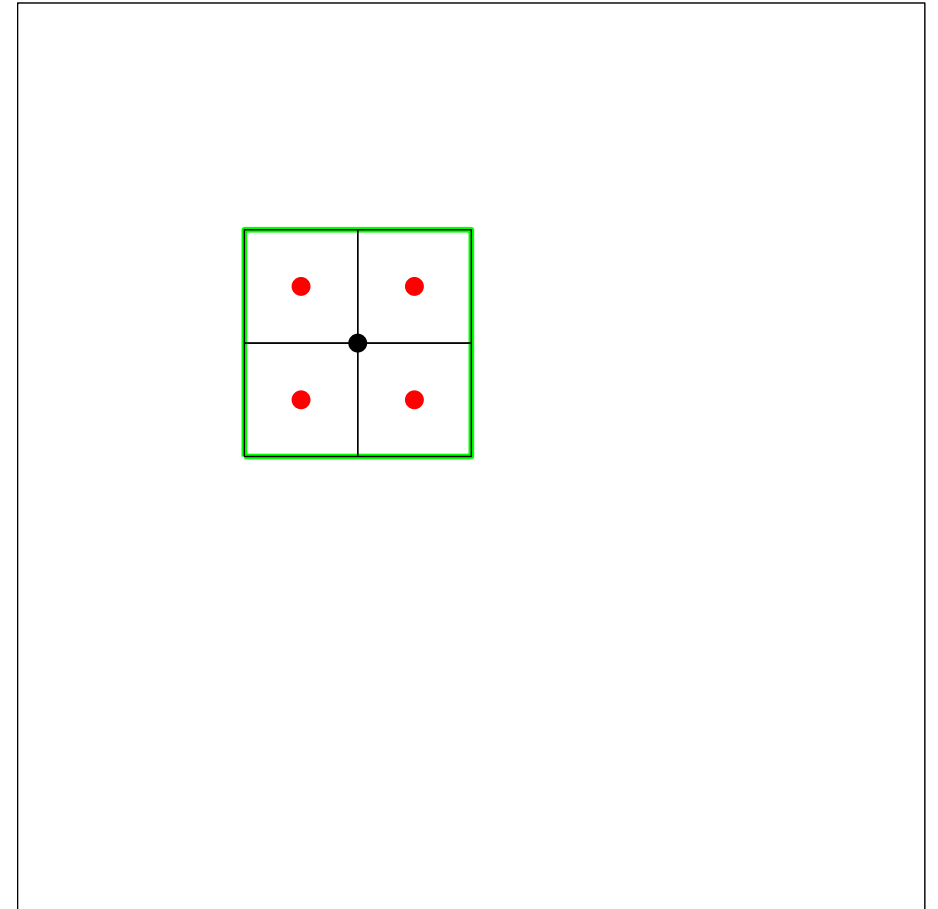
Let τ be a box (green).

Let \mathbf{c}_τ be the center of τ (black).

Let $\mathcal{L}_\tau^{(\text{child})}$ denote the children of τ .

Let \mathbf{c}_σ be the center of child σ .

Let $\hat{\mathbf{q}}_\sigma$ be the outgoing expansion of child σ .

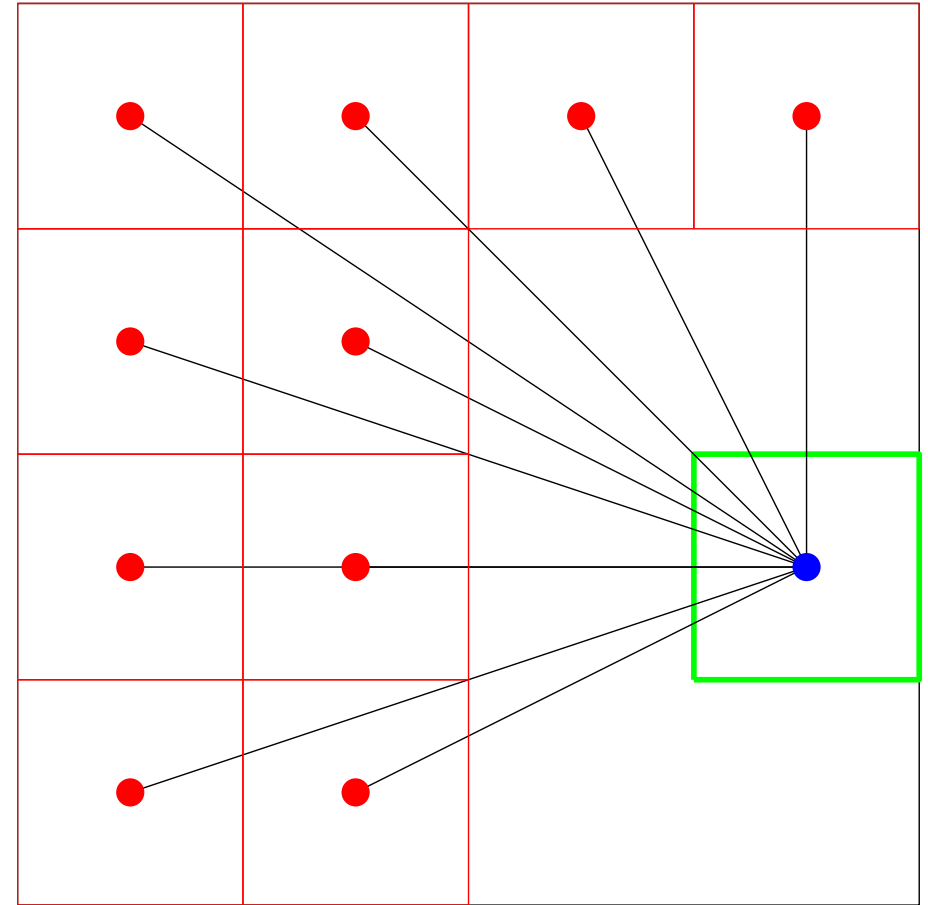


The outgoing expansion of τ can be computed from the outgoing expansions of its children:

$$\hat{\mathbf{q}}_\tau = \sum_{\sigma \in \mathcal{L}_\tau^{(\text{child})}} \mathsf{T}_{\tau, \sigma}^{(\text{of}_o)} \hat{\mathbf{q}}_\sigma.$$

Computing the incoming expansions on level 2

Let τ be a box on level 2 (green).
Let \mathbf{c}_τ be the center of τ (black).
The well-separated boxes on level 2 are red.



The incoming expansion of τ is computed from the outgoing expansions of boxes in its interaction list

$$\hat{\mathbf{u}}_\tau = \sum_{\sigma \in \mathcal{L}_\tau^{(\text{int})}} \mathsf{T}_{\tau, \sigma}^{(\text{ifo})} \hat{\mathbf{q}}_\sigma.$$

Computing the incoming expansions on level ℓ when $\ell > 2$

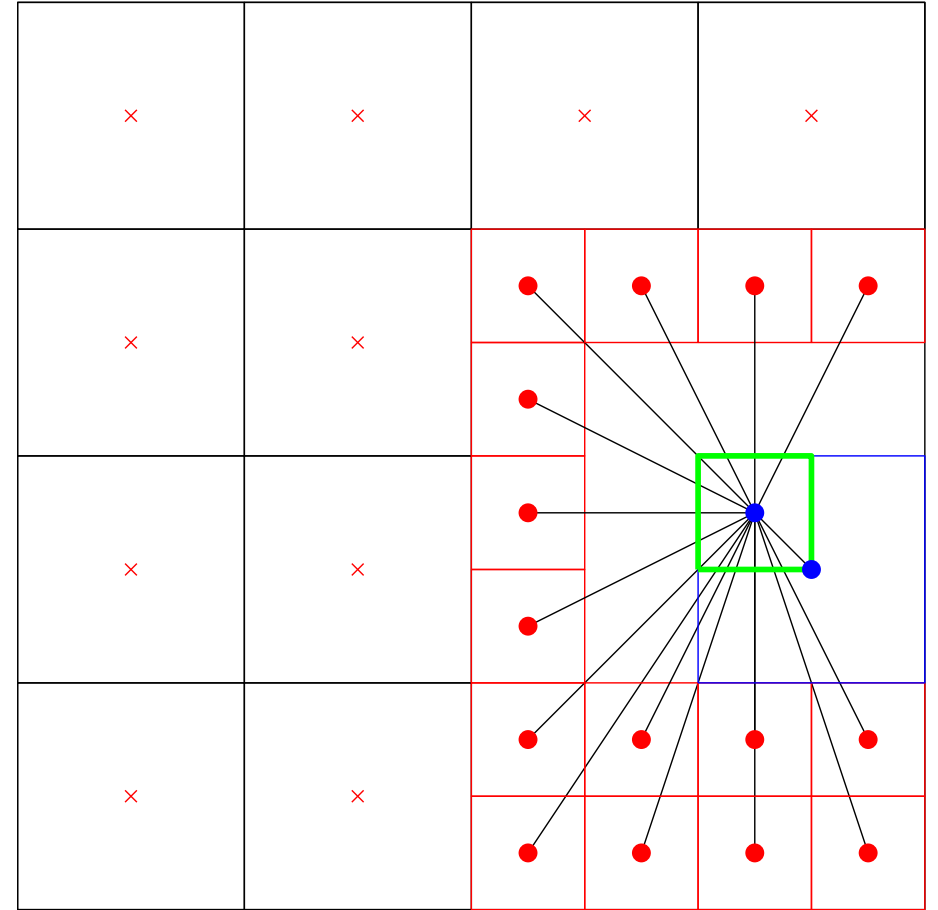
Let τ be a box on level $\ell = 3$ (green).

Let ν be the parent of τ (blue).

Let u_{in}^τ denote the potential caused by charges that are well-separated from τ — these are charges in the boxes marked with red dots and crosses. We have

$$u_{\text{in}}^\tau = u_{\text{in}}^\nu + v,$$

where u_{in}^ν is the incoming field for τ 's parent (caused by the boxes with red crosses), and v is the field caused by boxes in the interaction list of τ (boxes with a red dot).



The field u_{in}^ν was computed on the previous level and is represented by \hat{u}_ν .

The field v is computed by transferring the outgoing expansions \hat{q}_σ for $\sigma \in \mathcal{L}_\tau^{(\text{int})}$.

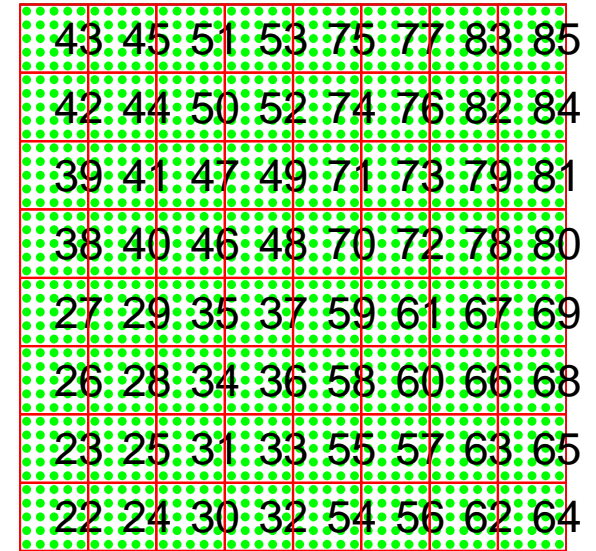
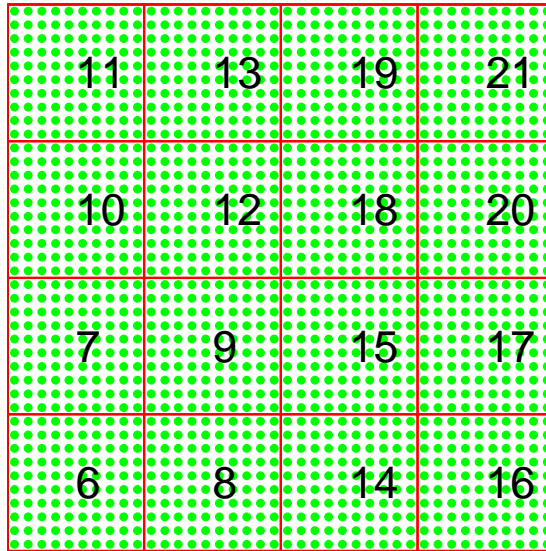
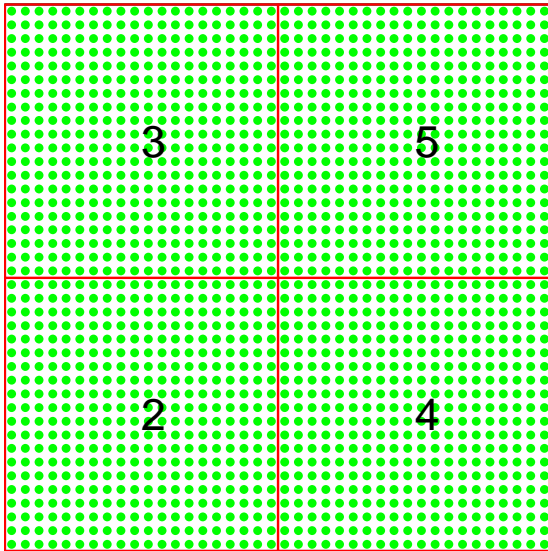
$$\hat{u}_\tau = T_{\tau,\nu}^{(\text{ifi})} \hat{u}_\nu + \sum_{\sigma \in \mathcal{L}_\tau^{(\text{int})}} T_{\tau,\sigma}^{(\text{ifo})} \hat{q}_\sigma$$

$\sim u_{\text{in}}^\tau$
 $\sim u_{\text{in}}^\nu$
 $\sim v$

THE CLASSICAL FAST MULTIPOLE METHOD IN \mathbb{R}^2

1. Construct the tree and all “interaction lists.”
2. For each leaf node, compute its outgoing expansion directly from the charges in the box via the *outgoing-from-sources operator*.
3. For each parent node, compute its outgoing expansion by merging the expansions of its children via the *outgoing-from-outgoing operator*.
4. For each node, compute its incoming expansion by transferring the incoming expansion of its parent (via the *incoming-from-incoming operator*), and then add the contributions from all charges in its interaction list (via the *incoming-from-outgoing operator*).
5. For each leaf node, evaluate the incoming expansion at the targets (via the *targets-from-incoming operator*), and compute near-field interactions directly.

Construct the tree and all interaction lists.



Let L denote the number of levels in the tree.

Set all potentials to zero:

For all boxes τ

$$\hat{\mathbf{u}}_\tau = \mathbf{0}$$

$$\hat{\mathbf{q}}_\tau = \mathbf{0}.$$

Set the potential to zero:

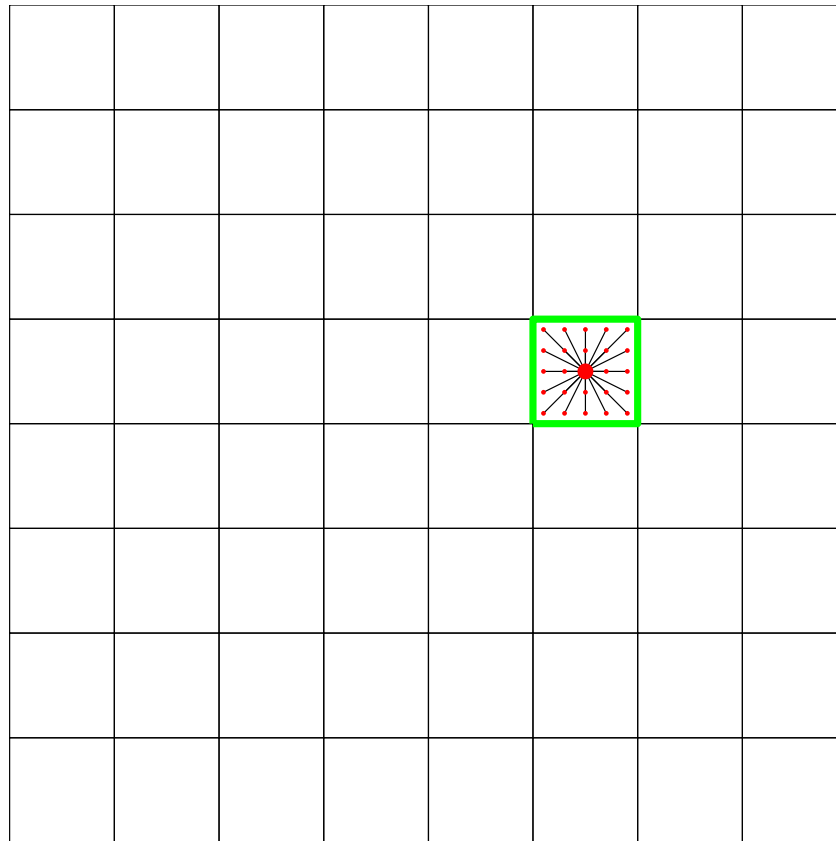
$$\mathbf{u} = \mathbf{0}.$$

Compute the outgoing expansion on each leaf via application of the *outgoing-from-source operators*:

loop over all leaf nodes τ

$$\hat{\mathbf{q}}_{\tau} = \mathsf{T}_{\tau}^{(\text{ofs})} \mathbf{q}(J_{\tau})$$

end loop



Compute the outgoing expansion of each parent by merging the expansions of its children via application of the *outgoing-from-outgoing operators*:

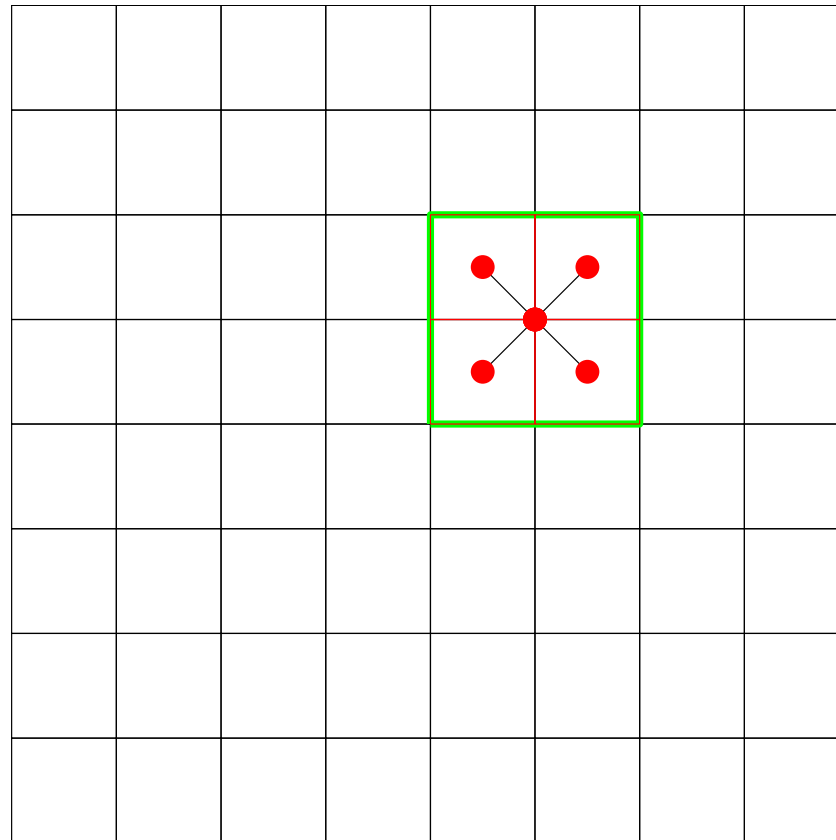
loop over levels $\ell = L - 1, L - 2, \dots, 2$

loop over all nodes τ on level ℓ

$$\hat{q}_\tau = \sum_{\sigma \in \mathcal{L}_\tau^{(\text{child})}} \mathbf{T}_{\tau, \sigma}^{(\text{of})} \hat{q}_\sigma$$

end loop

end loop

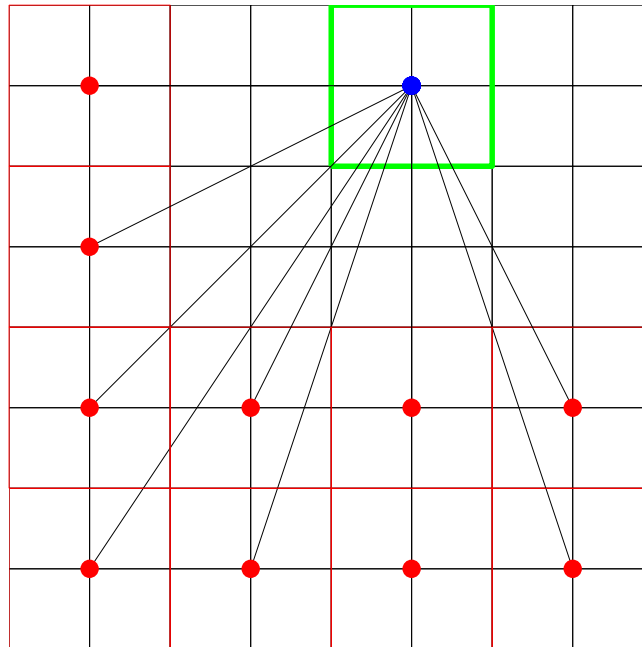


Add contributions from boxes in the interaction list of each box via the *incoming-from-outgoing operators*:

loop over all nodes τ

$$\hat{\mathbf{u}}_{\tau} = \hat{\mathbf{u}}_{\tau} + \sum_{\sigma \in \mathcal{L}_{\tau}^{(\text{int})}} \mathbf{T}_{\tau, \sigma}^{(\text{ifo})} \hat{\mathbf{q}}_{\sigma}.$$

end loop

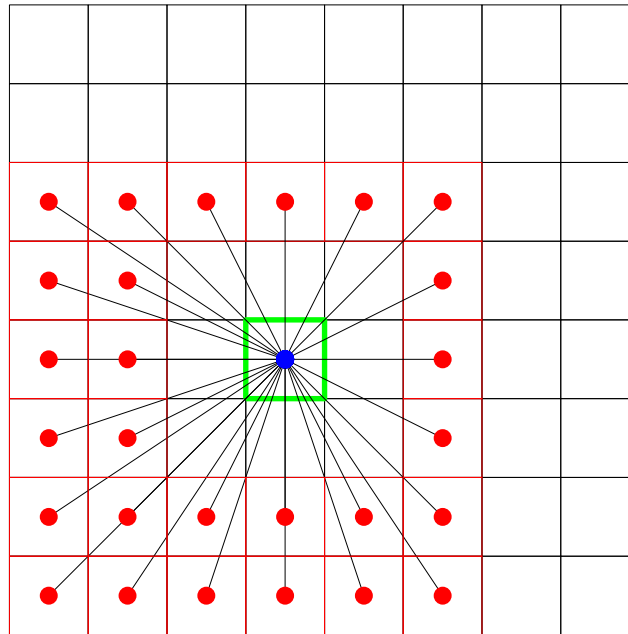


Add contributions from boxes in the interaction list of each box via the *incoming-from-outgoing operators*:

loop over all nodes τ

$$\hat{\mathbf{u}}_{\tau} = \hat{\mathbf{u}}_{\tau} + \sum_{\sigma \in \mathcal{L}_{\tau}^{(\text{int})}} \mathbf{T}_{\tau, \sigma}^{(\text{ifo})} \hat{\mathbf{q}}_{\sigma}.$$

end loop



Add contributions from the parent of each box via via the *incoming-from-incoming operators*:

loop over levels $\ell = 2, 3, 4, \dots, L - 1$

loop over all nodes τ on level ℓ

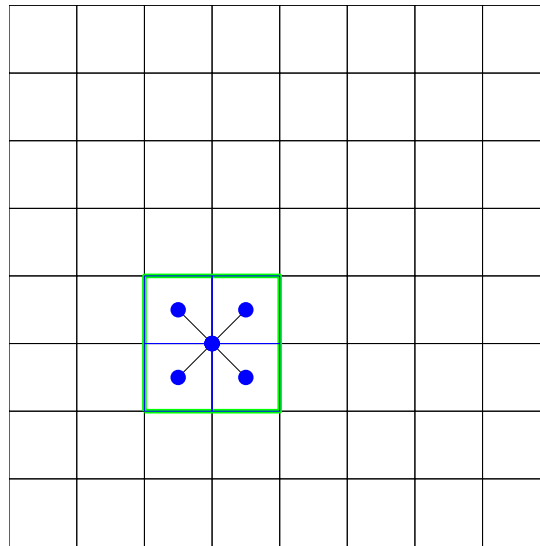
loop over all children σ of τ

$$\hat{u}_\sigma = \hat{u}_\sigma + \mathsf{T}_{\sigma,\tau}^{(\text{ifi})} \hat{u}_\tau.$$

end loop

end loop

end loop

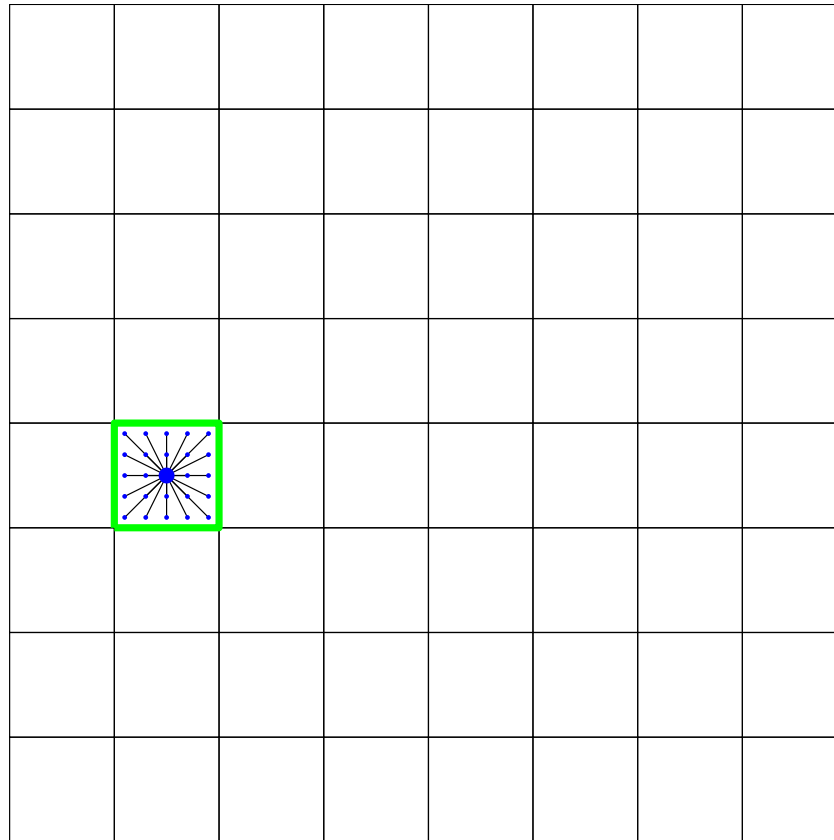


Compute the potential on every leaf by expanding its incoming potential via the *targets-from-incoming operators*:

loop over all leaf nodes τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \mathbf{T}_\tau^{(\text{tfi})} \hat{\mathbf{u}}_\tau$$

end loop

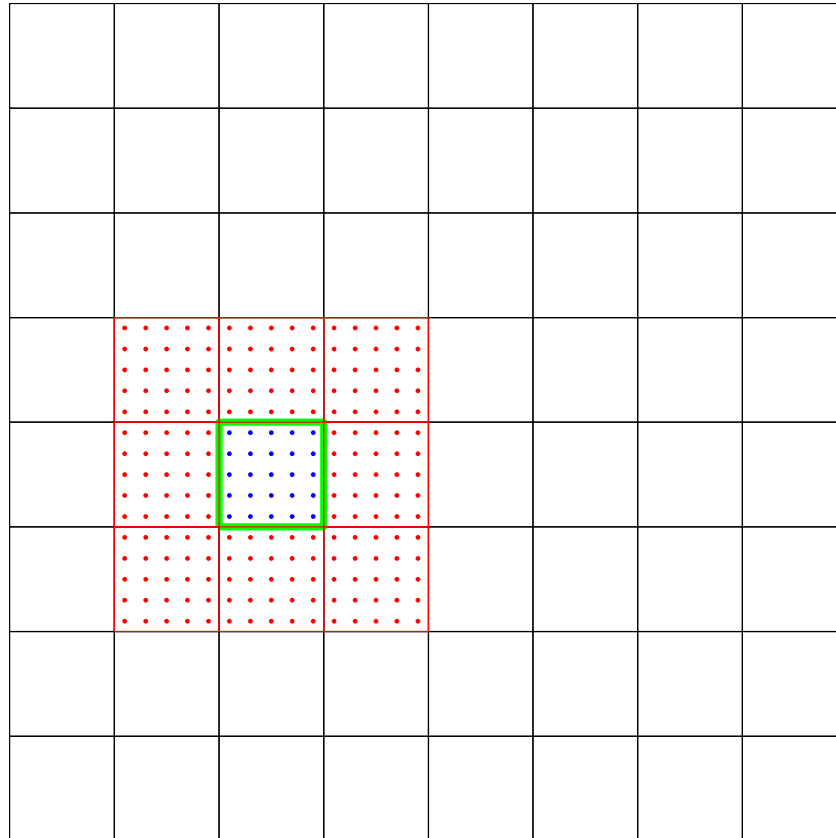


Add to the leaf potentials the interactions from direct neighbors:

loop over all leaf nodes τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \mathbf{A}(J_\tau, J_\tau) \mathbf{q}(J_\tau) + \sum_{\sigma \in \mathcal{L}_\tau^{(\text{nei})}} \mathbf{A}(J_\tau, J_\sigma) \mathbf{q}(J_\sigma)$$

end loop



Set $\hat{\mathbf{u}}_\tau = \mathbf{0}$ and $\hat{\mathbf{q}}_\tau = \mathbf{0}$ for all τ .

loop over all leaf nodes τ

$$\hat{\mathbf{q}}_\tau = \mathsf{T}_\tau^{(\text{ofs})} \mathbf{q}(J_\tau)$$

end loop

loop over levels $\ell = L, L - 1, \dots, 2$

loop over all nodes τ on level ℓ

$$\hat{\mathbf{q}}_\tau = \sum_{\sigma \in \mathcal{L}_\tau^{(\text{child})}} \mathsf{T}_{\tau,\sigma}^{(\text{ofc})} \hat{\mathbf{q}}_\sigma$$

end loop

end loop

loop over all nodes τ

$$\hat{\mathbf{u}}_\tau = \hat{\mathbf{u}}_\tau + \sum_{\sigma \in \mathcal{L}_\tau^{(\text{int})}} \mathsf{T}_{\tau,\sigma}^{(\text{ifo})} \hat{\mathbf{q}}_\sigma.$$

end loop

loop over levels $\ell = 2, 3, 4, \dots, L - 1$

loop over all nodes τ on level ℓ

loop over all children σ of τ

$$\hat{\mathbf{u}}_\sigma = \hat{\mathbf{u}}_\sigma + \mathsf{T}_{\sigma,\tau}^{(\text{ifi})} \hat{\mathbf{u}}_\tau.$$

end loop

end loop

end loop

loop over all leaf nodes τ

$$\mathbf{u}(J_\tau) = \mathsf{T}_\tau^{(\text{tff})} \hat{\mathbf{u}}_\tau$$

end loop

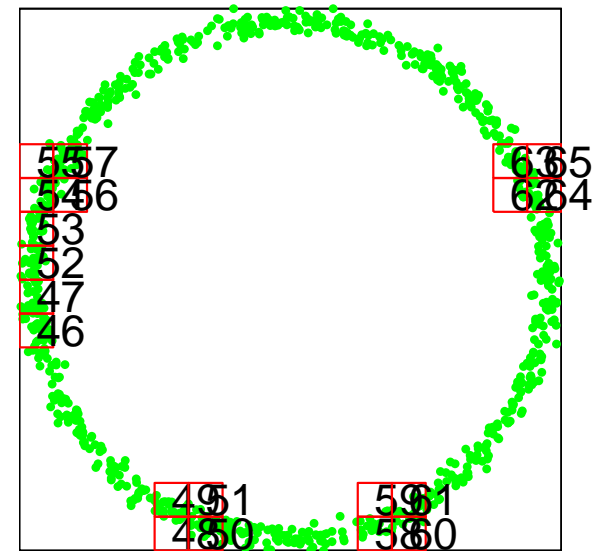
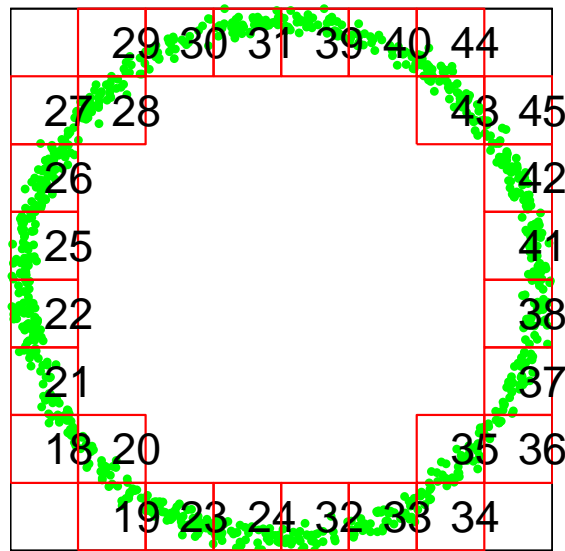
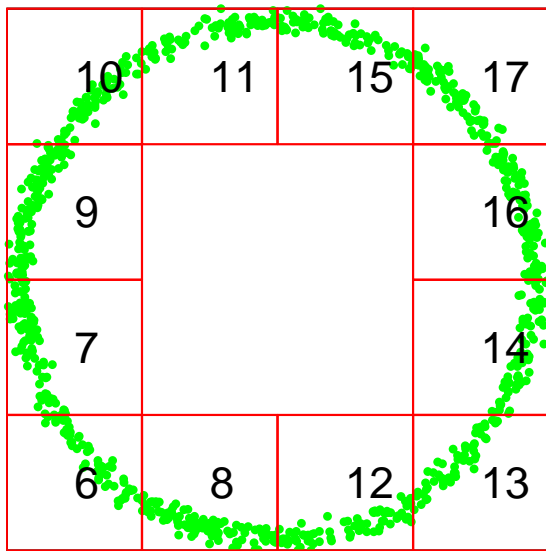
loop over all leaf nodes τ

$$\begin{aligned} \mathbf{u}(J_\tau) &= \mathbf{u}(J_\tau) + \mathbf{A}(J_\tau, J_\tau) \mathbf{q}(J_\tau) \\ &\quad + \sum_{\sigma \in \mathcal{L}_\tau^{(\text{nei})}} \mathbf{A}(J_\tau, J_\sigma) \mathbf{q}(J_\sigma) \end{aligned}$$

end loop

Now let us consider a non-uniform tree.

Construct the tree and all interaction lists.



Let L denote the number of levels in the tree.

Set all potentials to zero:

For all boxes τ

$$\hat{\mathbf{u}}_\tau = \mathbf{0}$$

$$\hat{\mathbf{q}}_\tau = \mathbf{0}.$$

Set the potential to zero:

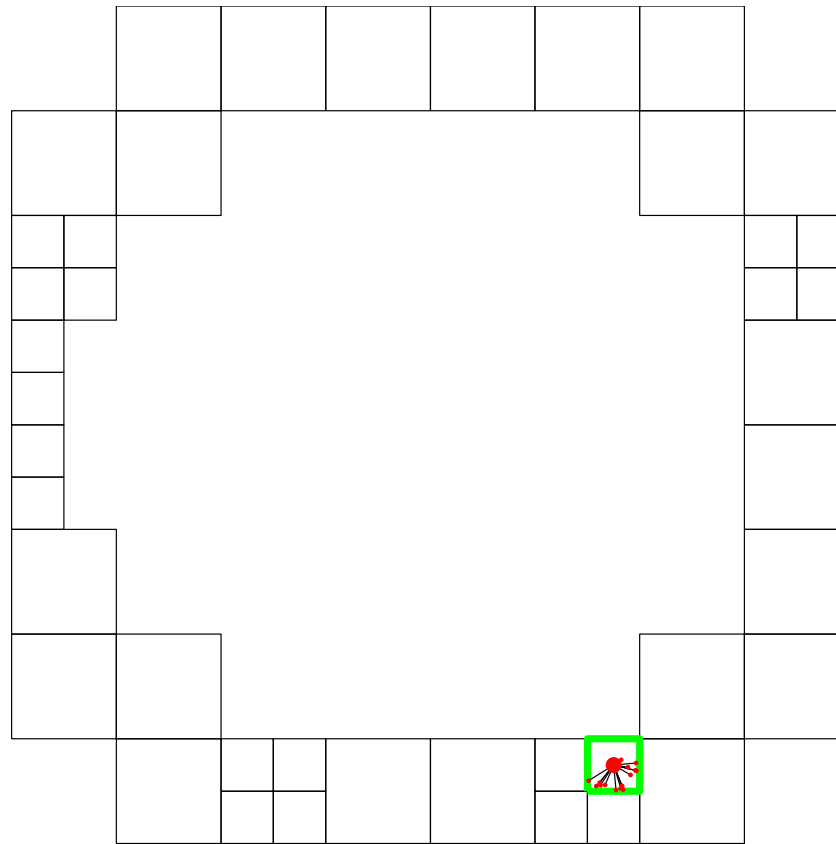
$$\mathbf{u} = \mathbf{0}.$$

Compute the outgoing expansion on each leaf via application of the *outgoing-from-source operators*:

loop over all leaf nodes τ

$$\hat{\mathbf{q}}_{\tau} = \mathsf{T}_{\tau}^{(\text{ofs})} \mathbf{q}(J_{\tau})$$

end loop

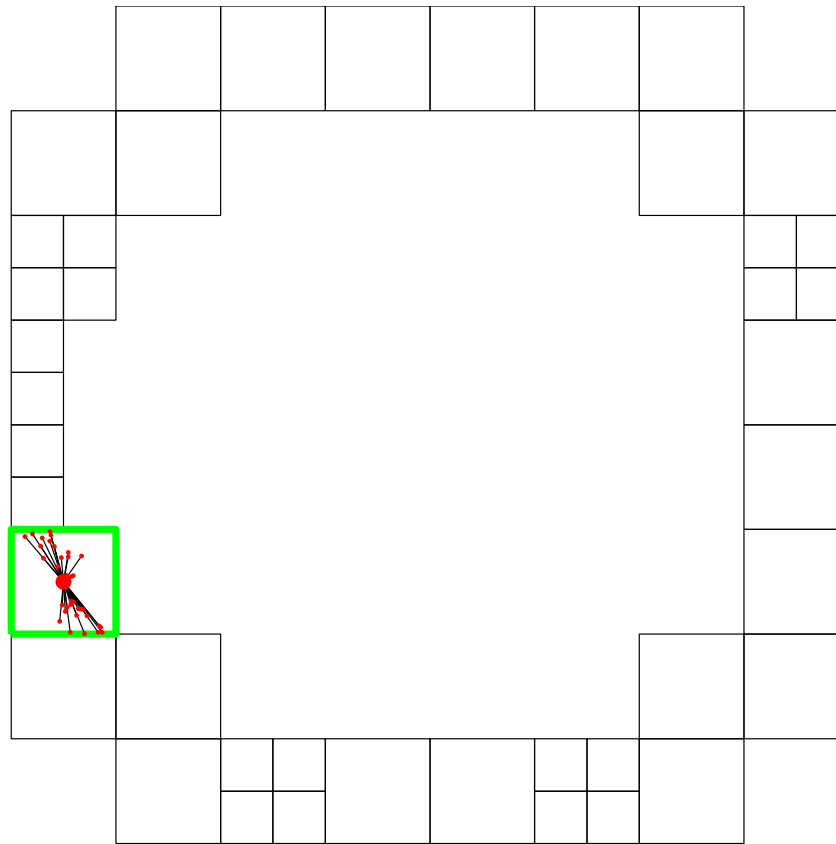


Compute the outgoing expansion on each leaf via application of the *outgoing-from-source operators*:

loop over all leaf nodes τ

$$\hat{\mathbf{q}}_{\tau} = \mathsf{T}_{\tau}^{(\text{ofs})} \mathbf{q}(J_{\tau})$$

end loop



Compute the outgoing expansion of each parent by merging the expansions of its children via application of the *outgoing-from-outgoing operators*:

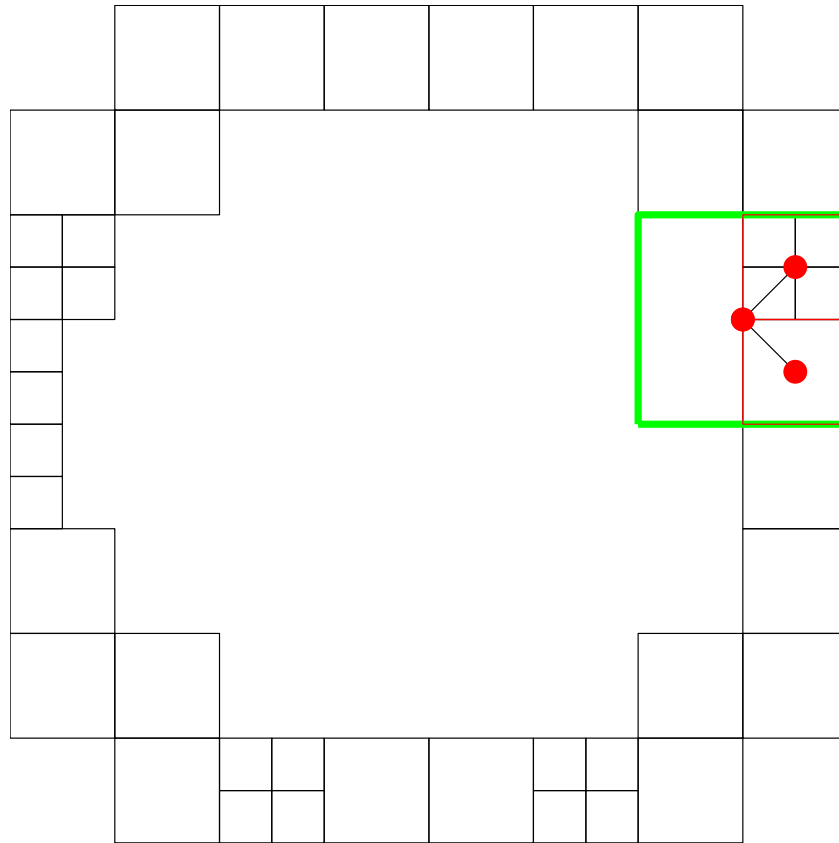
loop over levels $\ell = L - 1, L - 2, \dots, 2$

loop over all nodes τ on level ℓ

$$\hat{\mathbf{q}}_{\tau} = \sum_{\sigma \in \mathcal{L}_{\tau}^{(\text{child})}} \mathbf{T}_{\tau, \sigma}^{(\text{of})} \hat{\mathbf{q}}_{\sigma}$$

end loop

end loop



Add contributions from the parent of each box via via the *incoming-from-incoming operators*:

loop over levels $\ell = 2, 3, 4, \dots, L - 1$

loop over all nodes τ on level ℓ

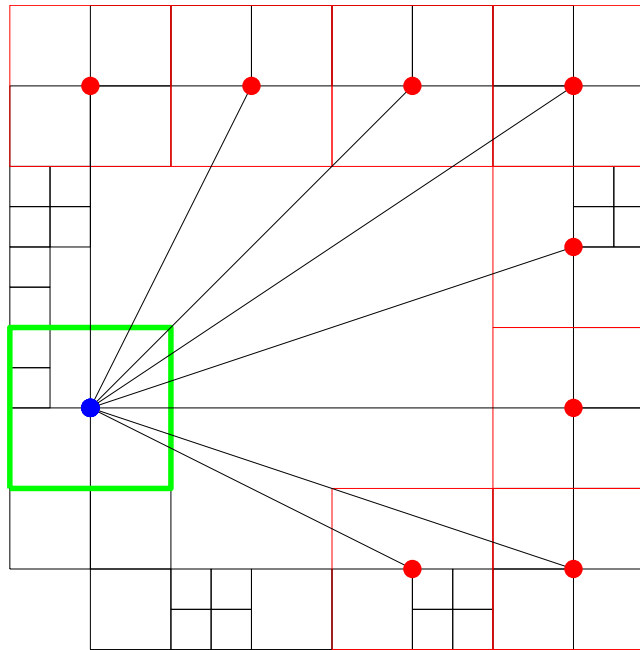
loop over all children σ of τ

$$\hat{u}_\sigma = \hat{u}_\sigma + \mathsf{T}_{\sigma,\tau}^{(\text{ifi})} \hat{u}_\tau.$$

end loop

end loop

end loop

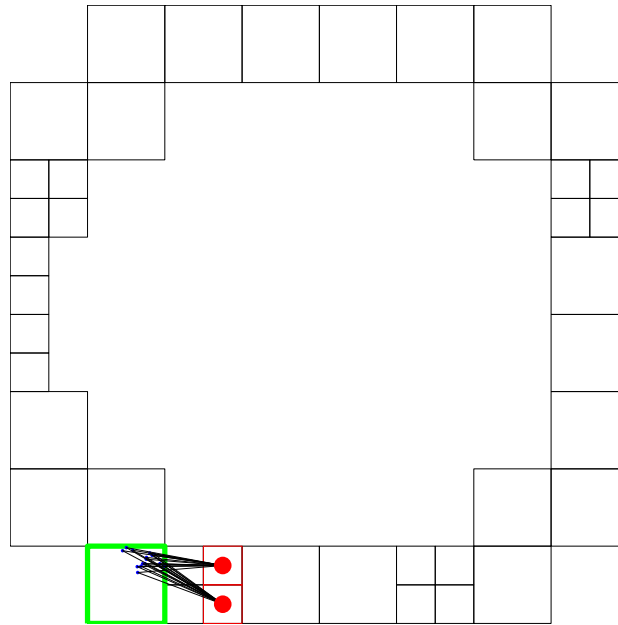


New: Some leaves τ (e.g. the green one below) must collect contributions to their potentials from outgoing expansions on boxes σ (red) on finer levels via the *targets-from-outgoing operator*:

loop over all nodes leaf τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \sum_{\sigma \in \mathcal{L}_\tau^{(3)}} \mathbf{T}_{\tau, \sigma}^{(\text{tfo})} \hat{\mathbf{q}}_\sigma.$$

end loop

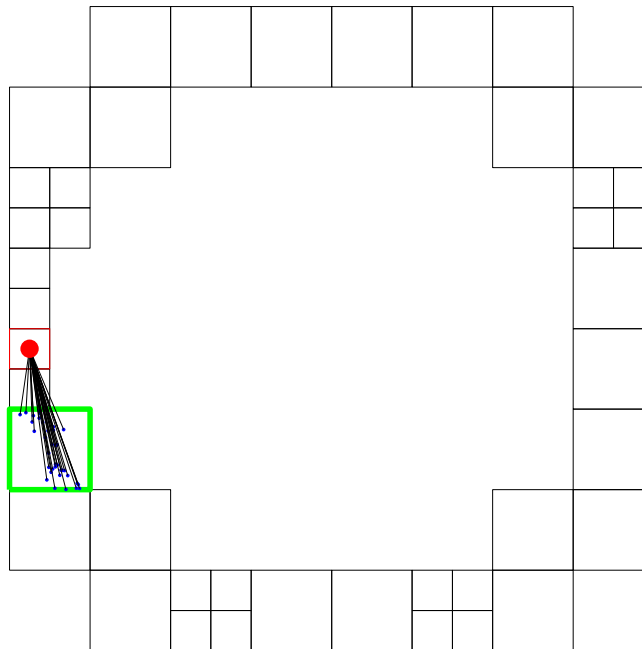


New: Some leaves τ (e.g. the green one below) must collect contributions to their potentials from outgoing expansions on boxes σ (red) on finer levels via the *targets-from-outgoing operator*:

loop over all nodes leaf τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \sum_{\sigma \in \mathcal{L}_\tau^{(3)}} \mathbf{T}_{\tau, \sigma}^{(\text{tfo})} \hat{\mathbf{q}}_\sigma.$$

end loop

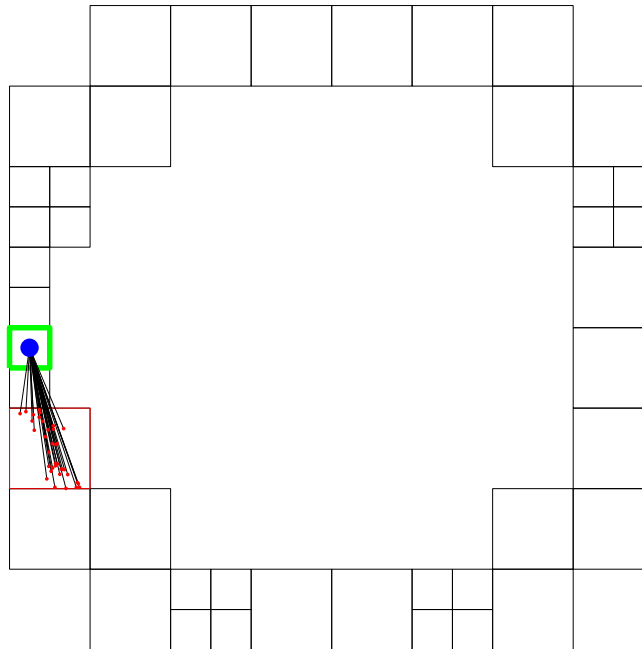


New: Some boxes τ (e.g. the green one) must collect contributions to their incoming expansions directly from the sources in some leaves σ (red) via the *incoming-from-sources operator*:

loop over all nodes τ

$$\hat{\mathbf{u}}_{\tau} = \hat{\mathbf{u}}_{\tau} + \sum_{\sigma \in \mathcal{L}_{\tau}^{(4)}} \mathbf{T}_{\tau, \sigma}^{(\text{ifs})} \mathbf{q}(J_{\sigma}).$$

end loop

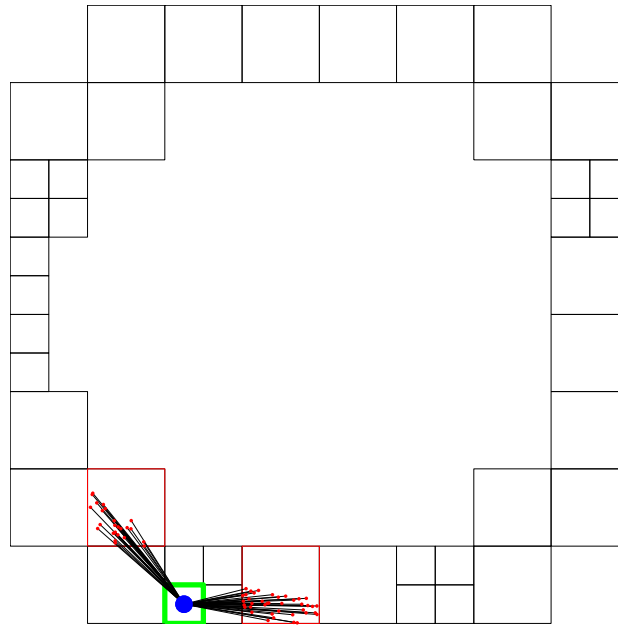


New: Some boxes τ (e.g. the green one) must collect contributions to their incoming expansions directly from the sources in some leaves σ (red) via the *incoming-from-sources operator*:

loop over all nodes τ

$$\hat{u}_\tau = \hat{u}_\tau + \sum_{\sigma \in \mathcal{L}_\tau^{(4)}} \mathsf{T}_{\tau, \sigma}^{(\text{ifs})} \mathbf{q}(J_\sigma).$$

end loop



Add contributions from the parent of each box via via the *incoming-from-incoming operators*:

loop over levels $\ell = 2, 3, 4, \dots, L - 1$

loop over all nodes τ on level ℓ

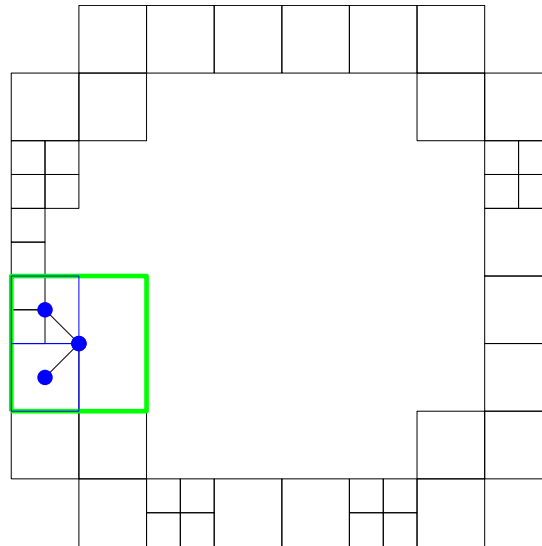
loop over all children σ of τ

$$\hat{u}_\sigma = \hat{u}_\sigma + \mathsf{T}_{\sigma,\tau}^{(\text{ifi})} \hat{u}_\tau.$$

end loop

end loop

end loop

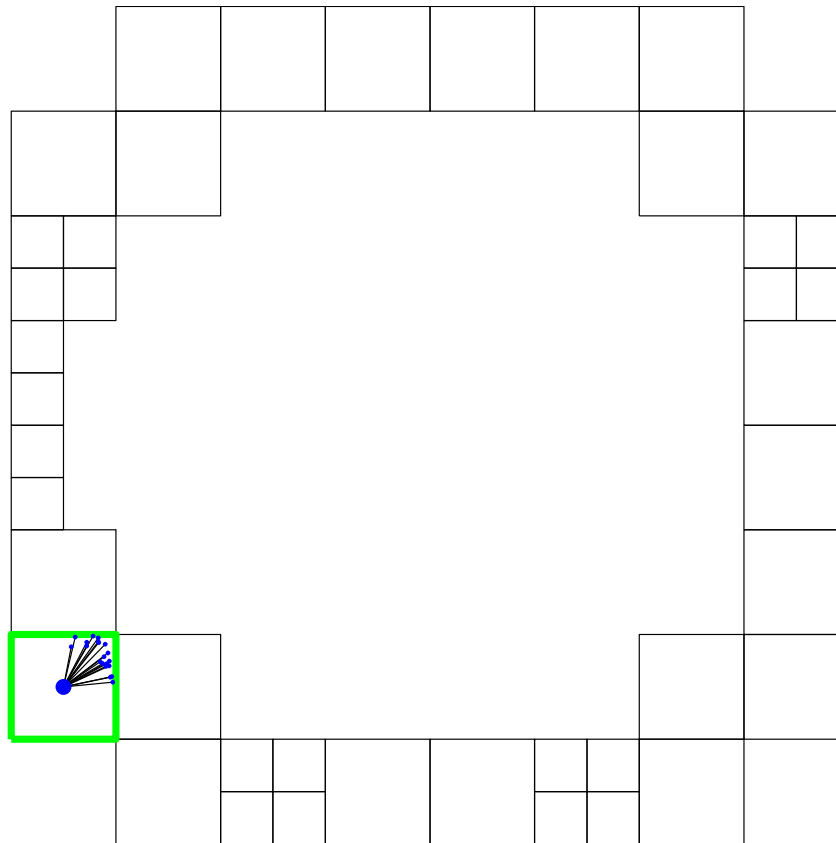


Compute the potential on every leaf by expanding its incoming potential via the *targets-from-incoming operators*:

loop over all leaf nodes τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \mathbf{T}_\tau^{(\text{tfi})} \hat{\mathbf{u}}_\tau$$

end loop

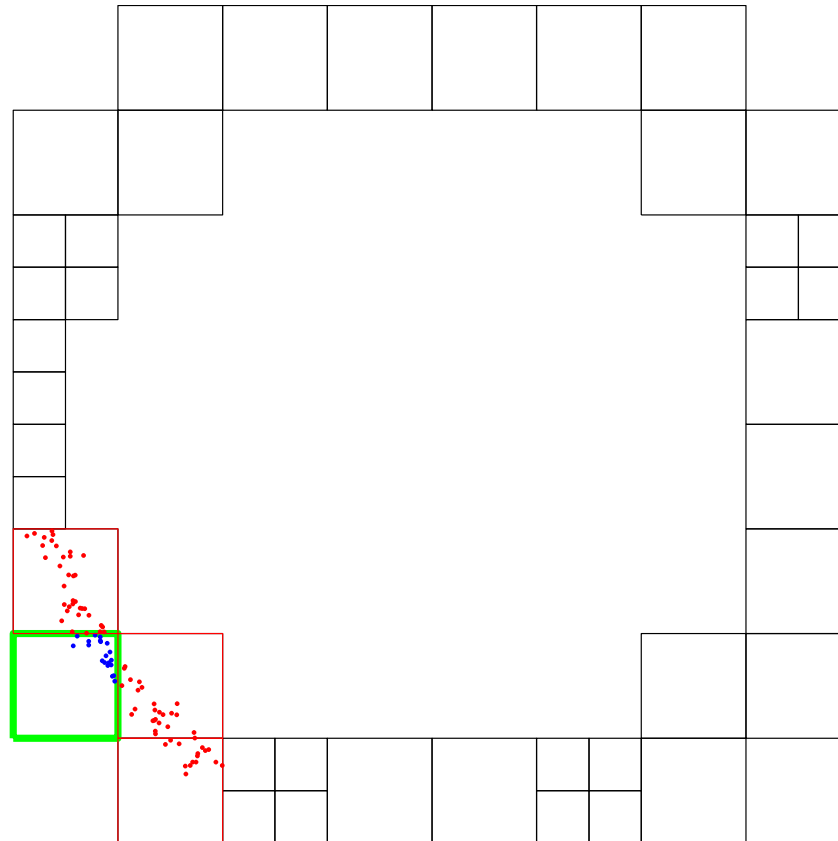


Add to the leaf potentials the interactions from direct neighbors:

loop over all leaf nodes τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \mathbf{A}(J_\tau, J_\tau) \mathbf{q}(J_\tau) + \sum_{\sigma \in \mathcal{L}_\tau^{(\text{nei})}} \mathbf{A}(J_\tau, J_\sigma) \mathbf{q}(J_\sigma)$$

end loop

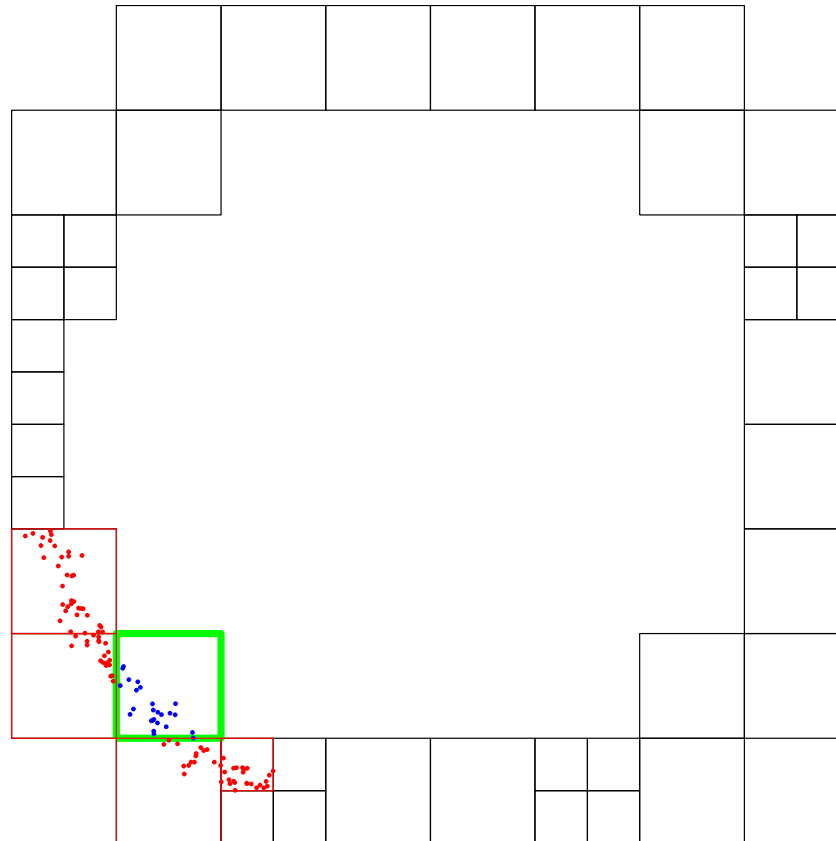


Add to the leaf potentials the interactions from direct neighbors:

loop over all leaf nodes τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \mathbf{A}(J_\tau, J_\tau) \mathbf{q}(J_\tau) + \sum_{\sigma \in \mathcal{L}_\tau^{(\text{nei})}} \mathbf{A}(J_\tau, J_\sigma) \mathbf{q}(J_\sigma)$$

end loop



Set $\hat{\mathbf{u}}_\tau = \mathbf{0}$ and $\hat{\mathbf{q}}_\tau = \mathbf{0}$ for all τ .

loop over all leaf nodes τ

$$\hat{\mathbf{q}}_\tau = \mathsf{T}_\tau^{(\text{ofs})} \mathbf{q}(J_\tau)$$

end loop

loop over levels $\ell = L, L - 1, \dots, 2$

loop over all nodes τ on level ℓ

$$\hat{\mathbf{q}}_\tau = \sum_{\sigma \in \mathcal{L}_\tau^{(\text{child})}} \mathsf{T}_{\tau,\sigma}^{(\text{ofc})} \hat{\mathbf{q}}_\sigma$$

end loop

end loop

loop over all nodes τ

$$\hat{\mathbf{u}}_\tau = \hat{\mathbf{u}}_\tau + \sum_{\sigma \in \mathcal{L}_\tau^{(\text{int})}} \mathsf{T}_{\tau,\sigma}^{(\text{ifo})} \hat{\mathbf{q}}_\sigma.$$

end loop

loop over all nodes τ

$$\hat{\mathbf{u}}_\tau = \hat{\mathbf{u}}_\tau + \sum_{\sigma \in \mathcal{L}_\tau^{(4)}} \mathsf{T}_{\tau,\sigma}^{(\text{ifs})} \mathbf{q}(J_\sigma).$$

end loop

loop over levels $\ell = 2, 3, 4, \dots, L - 1$

loop over all nodes τ on level ℓ

loop over all children σ of τ

$$\hat{\mathbf{u}}_\sigma = \hat{\mathbf{u}}_\sigma + \mathsf{T}_{\sigma,\tau}^{(\text{ifi})} \hat{\mathbf{u}}_\tau.$$

end loop

end loop

end loop

loop over all leaf nodes τ

$$\mathbf{u}(J_\tau) = \mathsf{T}_\tau^{(\text{tfi})} \hat{\mathbf{u}}_\tau$$

end loop

loop over all nodes τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \sum_{\sigma \in \mathcal{L}_\tau^{(3)}} \mathsf{T}_{\tau,\sigma}^{(\text{tfo})} \hat{\mathbf{q}}_\sigma.$$

end loop

loop over all leaf nodes τ

$$\begin{aligned} \mathbf{u}(J_\tau) &= \mathbf{u}(J_\tau) + \mathsf{A}(J_\tau, J_\tau) \mathbf{q}(J_\tau) \\ &\quad + \sum_{\sigma \in \mathcal{L}_\tau^{(\text{nei})}} \mathsf{A}(J_\tau, J_\sigma) \mathbf{q}(J_\sigma) \end{aligned}$$

end loop

A summary of the lists needed:

$\mathcal{L}_\tau^{(\text{child})}$ The children of τ .

$\mathcal{L}_\tau^{(\text{parent})}$ The parent of τ .

$\mathcal{L}_\tau^{(\text{nei})}$ For a leaf box τ , this is a list of the leaf boxes that directly border τ .
For a non-leaf box, $\mathcal{L}_\tau^{(\text{nei})}$ is empty.

$\mathcal{L}_\tau^{(\text{int})}$ A box $\sigma \in \mathcal{L}_\tau^{(\text{int})}$ iff σ and τ are on the same level,
 σ and τ are well-separated,
but the parents of σ and τ are not well-separated.

$\mathcal{L}_\tau^{(3)}$ For a *leaf* box τ , a box $\sigma \in \mathcal{L}_\tau^{(3)}$ iff σ lives on a finer level than τ ,
 τ is well-separated from σ , but τ is not well-separated from the parent of σ .
For a non-leaf box τ , $\mathcal{L}_\tau^{(3)}$ is empty.

$\mathcal{L}_\tau^{(4)}$ The dual of $\mathcal{L}_\tau^{(3)}$. In other words, $\sigma \in \mathcal{L}_\tau^{(4)}$ if and only if $\tau \in \mathcal{L}_\sigma^{(3)}$.

A summary of the translation operators:

