1. Let $\mathbf{A}$ be an $m \times n$ matrix, set $p = \min(m, n)$, and suppose that the singular value decomposition of $\mathbf{A}$ takes the form

$$\begin{array}{ccccc} \mathbf{A} & = & \mathbf{U} & \mathbf{D} & \mathbf{V}^* \\ m \times n & & m \times p & p \times p & p \times n. \end{array} \tag{1}$$

Let k be an integer such that $1 \leq k < p$ and let $\mathbf{A}_k$ denote the truncation of the SVD to the first k terms:

$$\mathbf{A}_k = \mathbf{U}(:, 1:k)\mathbf{D}(1:k, 1:k)\mathbf{V}(:, 1:k)^*.$$

Prove directly from the definition of the spectral and Frobenius norms that

$$\|\mathbf{A} - \mathbf{A}_k\| = \sigma_{k+1} \tag{2}$$

and that

$$\|\mathbf{A} - \mathbf{A}_k\|_F = \left(\sum_{j=k+1}^{p} \sigma_j^2 \right)^{1/2}. \tag{3}$$

Solution: First, partition the factorization $\mathbf{U}\mathbf{D}\mathbf{V}^*$ as

$$m \begin{array}{c} \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \\ \begin{array}{cc} k & p-k \end{array} \end{array} \begin{array}{c} k \\ p-k \end{array} \begin{array}{c} \begin{bmatrix} \mathbf{D}_1 & 0 \\ 0 & \mathbf{D}_2 \end{bmatrix} \\ \begin{array}{cc} k & p-k \end{array} \end{array} \begin{array}{c} k \\ p-k \end{array} \begin{array}{c} \begin{bmatrix} \mathbf{V}_1^* \\ \mathbf{V}_2^* \end{bmatrix} \\ n \end{array}.$$

Then observe that $\mathbf{U}_1 = \mathbf{U}(:, 1:k)$, $\mathbf{D}_1 = D(1:k, 1:k)$, and $\mathbf{V}_1^* = V(:, 1:k)^*$, so that $\mathbf{A}_k = \mathbf{U}_1\mathbf{D}_1\mathbf{V}_1^*$. By carrying out block multiplication on the partitioned factorization, we see that

$$\mathbf{A} = \mathbf{U}_1\mathbf{D}_1\mathbf{V}_1^* + \mathbf{U}_2\mathbf{D}_2\mathbf{V}_2^* = \mathbf{A}_k + \mathbf{U}_2\mathbf{D}_2\mathbf{V}_2^*,$$

so

$$\mathbf{A} - \mathbf{A}_k = \mathbf{U}_2\mathbf{D}_2\mathbf{V}_2^*. \tag{4}$$

(a) First we'll show that $\|\mathbf{A} - \mathbf{A}_k\| = \sigma_{k+1}$. Let $\mathbf{x} \in \mathbb{R}^n$ be any vector such that $\|\mathbf{x}\| = 1$. We will show that $\|(\mathbf{A} - \mathbf{A}_k)\mathbf{x}\| \leq \sigma_{k+1}$. We establish the notation that $\mathbf{v}_i$ and $\mathbf{u}_i$ are the columns of $\mathbf{V}$ and $\mathbf{U}$, respectively. Since the columns of $\mathbf{V}$ are orthonormal we can construct an orthonormal basis of $\mathbb{R}^n$: $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}, \dots, \mathbf{v}_n\}$ (note that vectors $\mathbf{v}_{p+1}$ through $\mathbf{v}_n$

are not actually columns of $\mathbf{V}$ but are simply used to construct the basis), and thus

$$\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{v}_n$$

for some $c_i, i = 1, \ldots, n$. Now, we have that

$$(\mathbf{A} - \mathbf{A}_k)\mathbf{x} = \mathbf{U}_2 \mathbf{D}_2 \mathbf{V}_2^* \mathbf{x}.$$

Since the i-th entry of $\mathbf{V}_2^* \mathbf{x}$ is $\langle \mathbf{v}_i, \mathbf{x} \rangle$, and since $\mathbf{x}$ is a linear combination of the orthonormal basis $\{\mathbf{v}_i\}_{i=1}^{n}$,

$$\mathbf{V}_2^* \mathbf{x} = \begin{bmatrix} c_{k+1} \\ c_{k+2} \\ \vdots \\ c_p \end{bmatrix},$$

and so

$$\mathbf{U}_2 \mathbf{D}_2 \mathbf{V}_2^* \mathbf{x} = \sum_{i=k+1}^{p} c_i \sigma_i \mathbf{u}_i$$

which implies

$$\|\mathbf{U}_2 \mathbf{D}_2 \mathbf{V}_2^* \mathbf{x}\| \leq \sigma_{k+1} \| \sum_{i=k+1}^{p} c_i \mathbf{u}_i \|.$$

Finally, by the orthonormality of $\{\mathbf{u}_i\}_{i=1}^{p}$,

$$\| \sum_{i=k+1}^{p} c_i \mathbf{u}_i \|^2 = \sum_{i=k+1}^{p} c_i^2,$$

and by the orthonormality of $\{\mathbf{v}_i\}_{i=1}^{n}$,

$$1 = \|\mathbf{x}\|^2 = \| \sum_{i=1}^{n} c_i \mathbf{v}_n \|^2 = \sum_{i=1}^{n} c_i^2 \implies \sum_{i=k+1}^{p} c_i^2 \leq 1,$$

and therefore

$$\|(\mathbf{A} - \mathbf{A}_k)\mathbf{x}\| \leq \sigma_{k+1} \| \sum_{i=k+1}^{p} c_i \mathbf{u}_i \| \leq \sigma_{k+1}.$$

Thus, we have shown that $\|\mathbf{A} - \mathbf{A}_k\| \leq \sigma_{k+1}$. Next, we observe that for $\mathbf{x} = \mathbf{v}_{k+1}$,

$$\|(\mathbf{A} - \mathbf{A}_k)\mathbf{x}\| = \|\sigma_{k+1} \mathbf{u}_{k+1}\| = \sigma_{k+1} \|\mathbf{u}_{k+1}\| = \sigma_{k+1},$$

so since $\|\mathbf{v}_{k+1}\| = 1$, $\|\mathbf{A} - \mathbf{A}_k\| \geq \sigma_{k+1}$. Therefore,

$$\|\mathbf{A} - \mathbf{A}_k\| = \sigma_{k+1}.$$

(b) Next, we'll prove (3). We have that $\|\mathbf{A} - \mathbf{A}_k\|_F = \|\mathbf{U}_2\mathbf{D}_2\mathbf{V}_2^*\|_F$, and we claim that $\|\mathbf{U}_2\mathbf{D}_2\mathbf{V}_2^*\|_F = \|\mathbf{D}_2\|_F$. Let $\mathbf{y}_i, i = 1, 2, \ldots, n$ be the columns of $\mathbf{D}_2\mathbf{V}_2^*$. Then we have

$$\|\mathbf{U}_2\mathbf{D}_2\mathbf{V}_2^*\|_F^2 = \sum_{i=1}^{n} \|\mathbf{U}_2\mathbf{y}_i\|_2^2 = \sum_{i=1}^{n} \langle \mathbf{U}_2\mathbf{y}_i, \mathbf{U}_2\mathbf{y}_i \rangle = \sum_{i=1}^{n} \|\mathbf{y}_i\|_2^2 = \|\mathbf{D}_2\mathbf{V}_2^*\|_F^2$$

by the orthonormality of $\mathbf{U}_2$, which implies

$$\|\mathbf{U}_2\mathbf{D}_2\mathbf{V}_2^*\|_F = \|\mathbf{D}_2\mathbf{V}_2^*\|_F.$$

Similarly, since the columns of $\mathbf{V}_2$ are orthonormal, we have

$$\|\mathbf{D}_2\mathbf{V}_2^*\|_F = \|(\mathbf{D}_2\mathbf{V}_2^*)^*\|_F = \|\mathbf{V}_2\mathbf{D}_2\|_F = \|\mathbf{D}_2\|_F.$$

Then we can compute $\|\mathbf{D}_2\|_F$ directly to obtain

$$\|\mathbf{A} - \mathbf{A}_k\|_F = \|\mathbf{D}_2\|_F = \left(\sum_{j=k+1}^{p} \sigma_j^2 \right)^{1/2}.$$