APPM – 5720 Fast Algorithms for Big Data

Reference Homework Problem
HW: 4
Question: 2

Rishabh Raghavendran
Q2. \( A \) is an \( n \times n \) matrix

\[ A(i,j) > 0 \ \forall \ i,j \]

\[ \sum_{i=1}^{n} A(i,j) = 1 \ \forall j \]

(a) \[ \sum_{j=1}^{n} p(j) = 1 \]

\( p \) is a vector of non-negative numbers.

\[ y'P = AP \quad p_i \geq \sum_{j=1}^{n} p(j) = 1 \]

Let us start with a \( 2 \times 2 \) matrix

\[ p'(j) = A \cdot p(j) \]

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

we know, \( a + c = 1 \)

\( b + d = 1 \)

\( e = f = 1 \)

we must prove, \( n \times y = 1 \)

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} ae + bf \\ ce + df \end{pmatrix} \]

\[ n + y = ae + bf + ce + df \]

\[ y = e(a + c) + f(b + d) \]

\[ y = e + f \]
\[ n + y = e + f \]

We can show the same for a 3 \times 3 matrix and so on.

Thus, 
\[
\sum_{j} p(j) = \sum_{j} \sum_{i} A(c, j) p(j)
\]

\[
= \sum_{j} p(j) \sum_{i} A(c, j)
\]

\[
= \sum_{i} p(c) \cdot 1
\]

\[
= 1
\]

\[
\sum_{j} p(j) = 1
\]

Hence Proved.
(b) Prove that $A$ has an eigenvector corresponding to eigenvalue $= 1$.

We know that for an eigenvalue to exist, it must satisfy the following condition:

$$ A\mathbf{v} = \lambda \mathbf{v} $$

If $\lambda = 1$,

$$ A\mathbf{v} = \mathbf{v} $$

We have to prove there exists a $\mathbf{v}$ which satisfies this equation.

Let us take $A^T$.

Now, $A$ had dim $i, j \to$, $A^T$ will be $j, i$ and each $\sum$ row will sum to $1$.

Further, we know that a matrix and its transpose have same evals, hence, $\lambda$ the condition is satisfied for $A^T$, $\lambda = 1$ must be an eval of $A$ and hence an eval must exist.

$$ A^T \mathbf{v} = \mathbf{v} $$

Let us take a $3 \times 3$ matrix
\[
\begin{pmatrix}
\alpha & \beta & \gamma \\
\delta & \epsilon & \zeta \\
\eta & \iota & \xi
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta \\
\iota
\end{pmatrix}
=
\begin{pmatrix}
\xi \\
\eta \\
\iota
\end{pmatrix}
\]

We know that:
\[
\begin{align*}
\alpha + \beta + \gamma &= 1 \\
\delta + \epsilon + \zeta &= 1 \\
\eta + \iota + \xi &= 1
\end{align*}
\]

So let \( \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \)

\[
\mathbf{A} \mathbf{v} = \\
\begin{pmatrix}
\alpha & \beta & \gamma \\
\delta & \epsilon & \zeta \\
\eta & \iota & \xi
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
=
\begin{pmatrix}
\alpha + \beta + \gamma \\
\delta + \epsilon + \zeta \\
\eta + \iota + \xi
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
= \mathbf{v}
\]
we can generalize this to a matrix of any size.

If $A$ is of size $i,j$, we need to take $v$ of size $i,j$.

$$v = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \end{pmatrix}^T$$

$\therefore Av = v$

$\therefore \lambda = 1$ is an eval of $A$

$\Rightarrow \lambda = 1$ must be an eval of $A$

$\Rightarrow A$ must have an eigenvector with corresponding eval $= 1$

Hence Proved.