1. REVIEW: SINGLE-PASS ALGORITHM FOR HERMITIAN MATRICES

This section is a review from class on 02/01/2016. Let $A$ be an $n \times n$ Hermitian matrix. Further, define $k$ to be our target rank and $p$ the oversampling parameter. For notational convenience, let $l = k + p$.

1.1. Single-Pass Hermitian - Stage A.

1) Draw Gaussian matrix $G$ of size $n \times l$
2) Compute $Y = AG$, our sampling matrix.
3) Find $Q = \text{orth}(Y)$ via QR

Recall: $A \approx QQ^*AQQ^*$. Let $C = Q^*AQ$. Calculate the eigendecomposition of $C$ to find: $C = \hat{U}\hat{D}\hat{U}^*$. Then $A \approx QCQ^* = Q(\hat{U}\hat{D}\hat{U}^*)Q^*$. Set $U = Q\hat{U}$, $U^* = \hat{U}^*Q^*$ and $A \approx UDU^*$. We find $C$ by solving $C(Q^*G) = Q^*Y$ in the least squares sense making sure to enforce $C^* = C$. Now consider replacing step (3) with the calculation of an ‘econ’ SVD on $Y$. Let $Q$ contain the first $k$ left singular vectors of our factorization and proceed as normal. This will yield a substantially overdetermined system. The above procedure relies on $A$ being symmetric, how do we proceed if this is not the case?

2. SINGLE-PASS FOR GENERAL MATRIX

Let $A$ be a real or complex valued $m \times n$ matrix. Further, define $k$ to be our target rank and $p$ the oversampling parameter. For notational convenience, let $l = k + p$. We aim to retrieve an approximate SVD:

$$A \approx UDV^*$$

$m \times n$ $m \times k$ $k \times k$ $k \times n$

To begin, we will modify “Stage A” from Section 1.1 to output orthonormal matrices $Q_c$, $Q_r$ such that:

$$A \approx \begin{bmatrix} Q_c & Q_r^* \end{bmatrix} \begin{bmatrix} A & Q_r \end{bmatrix}$$

$m \times n$ $m \times k$ $k \times m$ $m \times n$ $n \times k$ $k \times n$

With $Q_c$ an approximate basis for the column space of $A$ and $Q_r$ an approximate basis for the row space of $A$. We will then set $C = Q_c^*AQ_r$ and proceed in the usual fashion. First let’s justify the means in which we aim to find $C$.

First, right multiply $C$ by $Q_c^*G_c$ to find: $CQ_c^*G_c = Q_c^*AQ_cG_c = Q_c^*AG_c = Q_c^*Y_c$. Similarly, left multiply $C$ by $G_r^*Q_c$ to find: $G_r^*Q_cC = G_r^*Q_cAQ_r = G_r^*AQ_r = Y_r^*Q_r$. Keep this in mind when we move to ”Stage B”.

2.1. Single-Pass General - Stage A.

1) Draw Gaussian matrices $G_c$, $G_r$ of size $n \times l$
2) Compute $Y_c = AG_c$, $Y_r = A^*G_r$
3) Find $[Q_c, \cdot ] = \text{svd}(Y_c, \text{econ})$, $[Q_r, \cdot ] = \text{svd}(Y_r, \text{econ})$
4) $Q_c = Q_c(:,1:k)$, $Q_r = Q_r(:,1:k)$

2.2. Single-Pass General - Stage B.

5) Determine a $k \times k$ matrix $C$ by solving:

$$\begin{bmatrix} C \ (Q_c^*G_c) \\
\end{bmatrix} \begin{bmatrix} k \times k \\
\end{bmatrix} \begin{bmatrix} \end{bmatrix} \begin{bmatrix} Q_c^*Y_c \ (Q_r^*Q_c) C \\
\end{bmatrix} \begin{bmatrix} k \times l \\
\end{bmatrix} \begin{bmatrix} k \times k \\
\end{bmatrix} \begin{bmatrix} l \times k \\
\end{bmatrix}$$

in the least squares sense. (Note: There are $2k^2$ equations for $k^2$ unknowns which represents a system that is very overdetermined.)

6) Compute SVD: $C = \hat{U}\hat{D}\hat{V}^*$
7) Set $U = Q_c\hat{U}$, $V = Q_r\hat{V}$

It should be noted that the General case reduces to the Hermitian case given a suitable matrix $A$. A natural follow up questions targets the reduction of asymptotic complexity. Can we reduce the FLOP count, say from $O(mnk)$, to $O(mn\log(k))$?
3. Reduction of Asymptotic Complexity

3.1. Review of RSVD. Let $A$ be a dense $m \times n$ matrix that fits in RAM, designate $k$, $p$, $l$ in the usual fashion. When computing the RSVD of $A$, there are two FLOP intensive steps that require $O(mnk)$ operations (Please see course notes from 1/29/2016 for more detail). We will first concentrate on accelerating the computation of $Y = AG$ with $G$ an $n \times l$ Gaussian matrix. To do so, consider replacing $G$ by a new random matrix, $\Omega$ with a few (seemingly contradictory) properties. These are:

- $\Omega$ has enough structure to ensure that $A\Omega$ can be evaluated in $(mn\log(k))$ flops.
- $\Omega$ is random enough to be reasonably certain that the columns of $Y = A\Omega$ approximately span the column space of $A$.

How can such an $\Omega$ be found? Are there any examples of one?

3.2. Example of $\Omega$: Let $F$ be the $n \times n$ DFT and note $F^*F = I$ ($F$ is called a "rotation"). Define $D$ to be diagonal with random entries and $S$ a subsampling matrix. Let $\Omega$ be:

$$
\Omega = D \begin{bmatrix} F \\ S^* \end{bmatrix} \\
n \times l \hspace{1cm} n \times n \hspace{1cm} n \times n \hspace{1cm} n \times l
$$

We are one step closer to the mythical $\Omega$. Further details in subsequent lectures.