The Interpolative Decomposition (ID)

Let $A$ be an $m \times n$ matrix of exact rank $k$. Then $A$ admits three “structure preserving” factorizations that sacrifice orthonormality to gain faithfulness to the original data:

(Column ID) \[ A_{m \times n} = C_{m \times k} Z_{k \times n} \]

(Row ID) \[ = X_{m \times k} R_{k \times n} \]

(Double-Sided ID) \[ = X_{m \times k} A_{\text{ske}l} Z_{k \times n} \]

where

- $I_s$ and $J_s$ are index vectors marking the chosen rows and columns, respectively.
- $C = A(\cdot, J_s)$ holds $k$ columns of $A$.
- $Z$ contains a $k \times k$ identity matrix, $Z(\cdot, J_s) = I_k$ and $\max_{i,j} |Z(i, j)| \leq 1$.
- $R = A(I_s, \cdot)$ is a set of $k$ rows of $A$.
- $X$ contains a $k \times k$ identity matrix, $X(I_s, \cdot) = I_k$ and $\max_{i,j} |X(i, j)| \leq 1$.
- $A_{\text{ske}l} = A(I_s, J_s)$

1. ADVANTAGES/DISADVANTAGES OF ID

1.1. Advantages.

- If $A$ is sparse, then so are $C$ and $R$.
- If $A$ is non-negative, then so are $C$ and $R$, etc.
- Data Interpretation
- Storage Efficient (Huge saving for sparse matrices, small saving for dense matrices)

1.2. Disadvantages.

- Your basis is no longer orthonormal.
- For matrices of approximate rank $k$, the ID can be less optimal than the SVD (approximation error is generally the same as the QR).

2. ID AND THE COLUMN-PIVOTED QR ARE CLOSELY RELATED

2.1. Column/Row ID. Let $A$ be $m \times n$ and $\text{rank}(A) = k < \min(m, n)$. Let’s compute the CPQR of $A$:

\[ A_{m \times n}(\cdot, J) = Q_{m \times k} R_{k \times n} \]

Partition $J = [J_s \quad J_r]$, where $s$ stand for the skeleton and $r$ stands for the residual. $J_s$ points to the $k$ pivot vectors that were chosen and $R = [R_{11} \quad R_{12}]$.

We rewrite (1) as:

\[ [A(\cdot, J_s) \quad A(\cdot, J_r)] = [Q_1 R_{11} \quad Q_1 R_{12}] \]

We now see that

\[ A(\cdot, J_s) = Q_1 R_{11} =: C \]
\[ A(\cdot, J_r) = Q_1 R_{12} = Q_1 R_{11} R_{11}^{-1} R_{12} = CT \]

where $T := R_{11}^{-1} R_{12}$ Write (1) as $AP = QR$ and then we get:

\[ A = Q_1 R P^* = Q_1 [R_{11} \quad R_{12}] P^* \]
\[ = Q_1 R_{11} [I_k R_{11}^{-1} \quad R_{12}] P^* = C [I_k \quad T] P^* = CZ \]
where $Z = [I_k \ T]P^*$.

Now consider a matrix of approximate rank $k$. Then:

$$\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$$

$\Rightarrow A = Q_1 [R_{11} \ R_{12}]P^* + Q_2 [0 \ R_{22}]P^*$

The first term can be written as:

$$Q_1 [R_{11} \ R_{12}]P^* = Q_1 R_{11} \ [I \ R_{11}^{-1}R_{12}]P^* = CZ$$

where $C = Q_1 R_{11}$ and $Z = [I \ R_{11}^{-1}R_{12}]P^*$ while the second term is the remainder term. Then:

$$A - CZ = Q_2 [0 \ R_{12}]P^*$$

which produces exactly the same error as in the truncated QR. In practice, just take $k$ steps of the Gram-Schmidt. Then:

$$\begin{bmatrix} R_{11} & R_{12} \\ 0 & \end{bmatrix}$$

Stop when $\|B\|_{\text{Fro}} \leq \epsilon$. Note that $\|B\| = \|Q_2 R_{22} \tilde{P}\| = \|R_{22}\|$.

To obtain the row ID, perform the Gram-Schmidt on the rows of $A$ instead of the columns (same as doing the QR factorization of $A^*$).

2.2. Double-Sided ID.

**Step 1:** Perform the column ID, $A \approx CZ$.

**Step 2:** Execute the row on $C$ (which is much smaller than $A$)! Since $C = XC(I_s, :)$ is an exact factorization and using $C = A(:, J_s)$, we have $C(I_s, :) = A(I_s, J_s)$. So:

$$A \approx CZ = XC(I_s, :)Z =XA(I_s, J_s)Z$$

where $A(I_s, J_s) = A_{\text{skel}}$.

You can also perform the Double-Sided ID using the row ID first (conduct the Gram-Schmidt on the smaller dimension for a more optimal solution).