The Interpolative Decomposition (ID)

Let $\mathbf{A}$ be an $m \times n$ matrix of exact rank k. Then $\mathbf{A}$ admits three "structure preserving" factorizations that sacrifice orthonormality to gain faithfulness to the original data:

(Column ID) $$\underset{m\times n}{\mathbf{A}} = \underset{m\times k}{\mathbf{C}}\ \underset{k\times n}{\mathbf{Z}}$$

(Row ID) $$= \underset{m\times k}{\mathbf{X}}\ \underset{k\times n}{\mathbf{R}}$$

(Double-Sided ID) $$= \underset{m\times k}{\mathbf{X}}\ \underset{k\times k}{\mathbf{A}_{\text{skel}}}\ \underset{k\times n}{\mathbf{Z}}$$

where

- I_s and J_s are index vectors marking the chosen rows and columns, respectively.
- $\mathbf{C} = \mathbf{A}(:, J_s)$ holds k columns of $\mathbf{A}$.
- $\mathbf{Z}$ contains a $k \times k$ identity matrix, $\mathbf{Z}(:, J_s) = \mathbf{I}_k$ and $\max_{i,j} |\mathbf{Z}(i, j)| \leq 1$.
- $\mathbf{R} = \mathbf{A}(I_s, :)$ is a set of k rows of $\mathbf{A}$.
- $\mathbf{X}$ contains a $k \times k$ identity matrix, $\mathbf{X}(I_s, :) = \mathbf{I}_k$ and $\max_{i,j} |\mathbf{X}i, j| \leq 1$.
- $\mathbf{A}_{\text{skel}} = \mathbf{A}(I_s, J_s)$

1. Advantages/Disadvantages of ID

1.1. Advantages.

- If $\mathbf{A}$ is sparse, then so are $\mathbf{C}$ and $\mathbf{R}$.
- If $\mathbf{A}$ is non-negative, then so are $\mathbf{C}$ and $\mathbf{R}$, etc.
- Data Interpretation
- Storage Efficient (Huge saving for sparse matrices, small saving for dense matrices)

1.2. Disadvantages.

- Your basis is no longer orthonormal.
- For matrices of approximate rank k, the ID can be less optimal than the SVD (approximation error is generally the same as the QR).

2. ID and the Column-Pivoted QR Are Closely Related

2.1. **Column/Row ID.** Let $\mathbf{A}$ be $m \times n$ and $\text{rank}(\mathbf{A}) = k < \min(m, n)$. Lets compute the CPQR of $\mathbf{A}$:

$$\underset{m\times n}{\mathbf{A}}(:, J) = \underset{m\times k}{\mathbf{Q}}\ \underset{k\times n}{\mathbf{R}} \tag{1}$$

Partition $J = [\underset{k}{J_s} \quad \underset{n-k}{J_r}]$ where s stand for the skeleton and r stands for the residual. J_s points to the k pivot vectors that were chosen and $\mathbf{R} = [\underset{k\times k}{\mathbf{R}_{11}} \quad \underset{k\times(n-k)}{\mathbf{R}_{12}}]$. We rewrite (1) as:

$$[\mathbf{A}(:, J_s) \quad \mathbf{A}(:, J_r)] = [\mathbf{Q}_1\mathbf{R}_{11} \quad \mathbf{Q}_1\mathbf{R}_{12}].$$

We now see that

$$\mathbf{A}(: J_s) = \mathbf{Q}_1\mathbf{R}_{11} =: \mathbf{C}$$
$$\mathbf{A}(:, J_r) = \mathbf{Q}_1\mathbf{R}_{12} = \mathbf{Q}_1\mathbf{R}_{11}\mathbf{R}_{11}^{-1}\mathbf{R}_{12} = \mathbf{CT}$$

where $\mathbf{T} := \mathbf{R}_{11}^{-1}\mathbf{R}_{12}$ Write (1) as $\mathbf{AP} = \mathbf{QR}$ and then we get:

$$\mathbf{A} = \mathbf{Q}_1\mathbf{R}\mathbf{P}^* = \mathbf{Q}_1 [\mathbf{R}_{11} \quad \mathbf{R}_{12}]\, \mathbf{P}^*$$
$$= \mathbf{QR}_{11} \left[\mathbf{I}_k \mathbf{R}_{11}^{-1} \quad \mathbf{R}_{12}\right] \mathbf{P}^* = \mathbf{C}[\mathbf{I}_k \quad \mathbf{T}]\, \mathbf{P}^* = \mathbf{CZ}$$

where $\mathbf{Z} = [\mathbf{I}_k \; \mathbf{T}]\mathbf{P}^*$.

Now consider a matrix of approximate rank k. Then:

$$\mathbf{AP} = [\mathbf{Q}_1 \;\; \mathbf{Q}_2] \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ 0 & \mathbf{R}_{22} \end{bmatrix}$$
$$\Rightarrow \mathbf{A} = \mathbf{Q}_1 [\mathbf{R}_{11} \;\; \mathbf{R}_{12}] \, \mathbf{P}^* + \mathbf{Q}_2 [0 \;\; \mathbf{R}_{22}] \, \mathbf{P}^*$$

The first term can be written as:

$$\mathbf{Q}_1 [\mathbf{R}_{11} \;\; \mathbf{R}_{12}] \, \mathbf{P}^* = \mathbf{Q}_1 \mathbf{R}_{11} \left[\mathbf{I} \;\; \mathbf{R}_{11}^{-1}\mathbf{R}_{12}\right] \mathbf{P}^* = \mathbf{CZ}$$

where $\mathbf{C} = \mathbf{Q}_1\mathbf{R}_{11}$ and $\mathbf{Z} = \left[\mathbf{I} \;\; \mathbf{R}_{11}^{-1}\mathbf{R}_{12}\right] \mathbf{P}^*$ while the second term is the remainder term. Then:

$$\mathbf{A} - \mathbf{CZ} = \mathbf{Q}_2 [0 \;\; \mathbf{R}_{12}] \, \mathbf{P}^*$$

which produces exactly the same error as in the truncated QR. In practice, just take k steps of the Gram-Schmidt. Then:

$$\mathbf{AP} = \mathbf{Q}_1 [\mathbf{R}_{11} \;\; \mathbf{R}_{12}] + [\underset{k}{0} \;\; \underset{n-k}{\mathbf{B}}]$$

Stop when $||\mathbf{B}||_{Fro} \leq \epsilon$. Note that $||\mathbf{B}|| = ||\mathbf{Q}_2\mathbf{R}_{22}\tilde{\mathbf{P}}|| = ||\mathbf{R}_{22}||$.

To obtain the row ID, perform the Gram-Schmidt on the <u>rows</u> of $\mathbf{A}$ instead of the columns (same as doing the QR factorization of $\mathbf{A}^*$).

2.2. **Double-Sided ID.**

Step 1: Perform the column ID, $\mathbf{A} \approx \mathbf{CZ}$.

Step 2: Execute the row ID on **C** (which is much smaller than **A**)! Since $\mathbf{C} = \mathbf{XC}(I_s, :)$ is an exact factorization and using $\mathbf{C} = \mathbf{A}(:, J_s)$, we have $\mathbf{C}(I_s, :) = \mathbf{A}(I_s, J_s)$. So:

$$\mathbf{A} \approx \mathbf{CZ} = \mathbf{XC}(I_s, :)\mathbf{Z} = \mathbf{XA}(I_s, J_s)\mathbf{Z}$$

where $\mathbf{A}(I_s, J_s) = \mathbf{A}_{\text{skel}}$.

You can also perform the Double-Sided ID using the row ID first (conduct the Gram-Schmidt on the smaller dimension for a more optimal solution).

REFERENCES