APPM 4720/5720 — week 13:

The Potential Evaluation Map

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The potential evaluation map

In this lecture, we will look more carefully at the map that given a source distribution q in a ("source") domain $\Omega_{\rm s}$ evaluates the potential in a ("target") domain $\Omega_{\rm t}$:

$$[Aq](\boldsymbol{x}) = \int_{\Omega_{\rm s}} \phi(\boldsymbol{x} - \boldsymbol{y})\, q(\boldsymbol{y})\, dA(\boldsymbol{y}), \qquad \boldsymbol{x} \in \Omega_{\rm t}.$$

We will cover two cases in detail:

1. Laplace: $\phi(\boldsymbol{x}) = \log|\boldsymbol{x}|$.
2. Helmholtz: $\phi(\boldsymbol{x}) = H_0^{(1)}(\kappa|\boldsymbol{x}|)$.

The discussion will more generally apply to elasticity, Stokes, the equations of elasticity, time-harmonic Maxwell, etc.

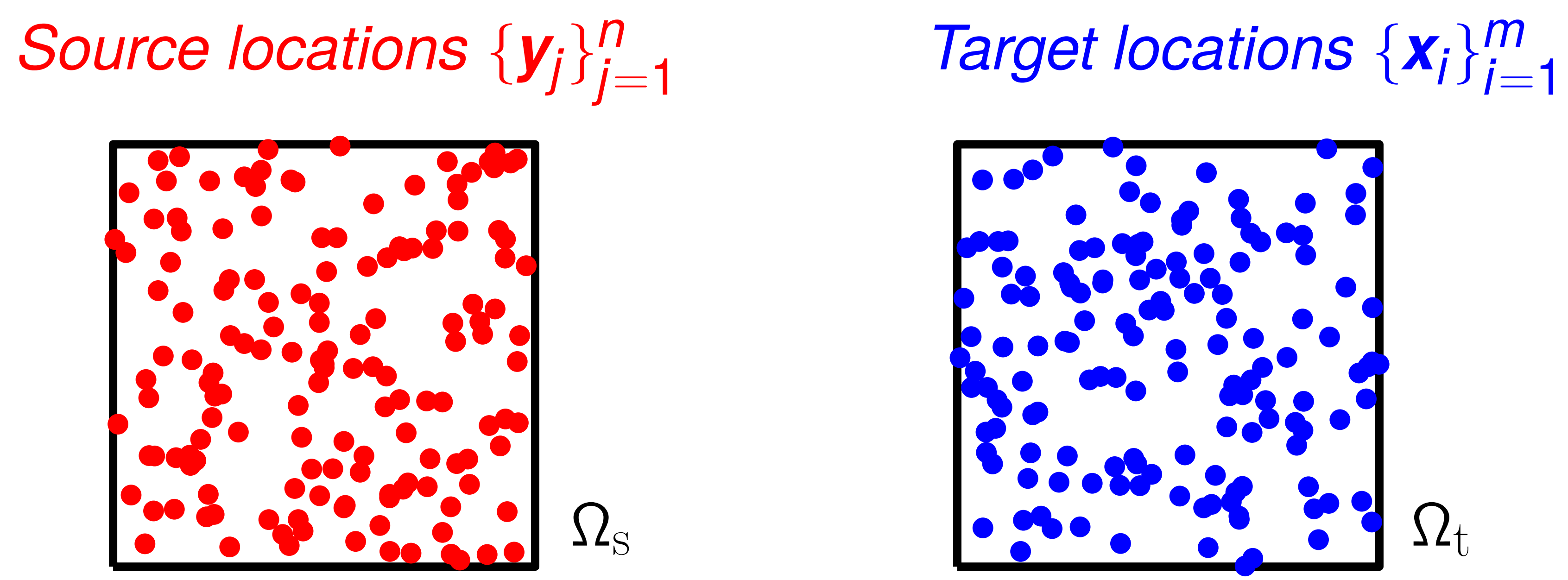
Themes:

- The effective "rank of interaction".
- Loss of information.
- Techniques for "compressing" the interaction.

Let us start with a Laplace problem (for me, it helps to think of it as electro-statics).

Suppose we are given two "well-separated" domains $\Omega_{\rm s}$ and $\Omega_{\rm t}$.

There are m sources in $\Omega_{\rm s}$ inducing n potentials in $\Omega_{\rm t}$.



Let **A** denote the $m \times n$ matrix with entries

$$\mathbf{A}(i,j) = \log|\boldsymbol{x}_i - \boldsymbol{y}_j|.$$

Given a vector $\mathbf{q} \in \mathbb{R}^n$ of source strengths, we seek a vector of potentials $\mathbf{f} \in \mathbb{R}^m$, where

$$\underset{m \times 1}{\mathbf{f}} = \underset{m \times n}{\mathbf{A}} \; \underset{n \times 1}{\mathbf{q}}$$

Using direct evaluation, the cost is $O(mn)$.

A is the $m \times n$ matrix

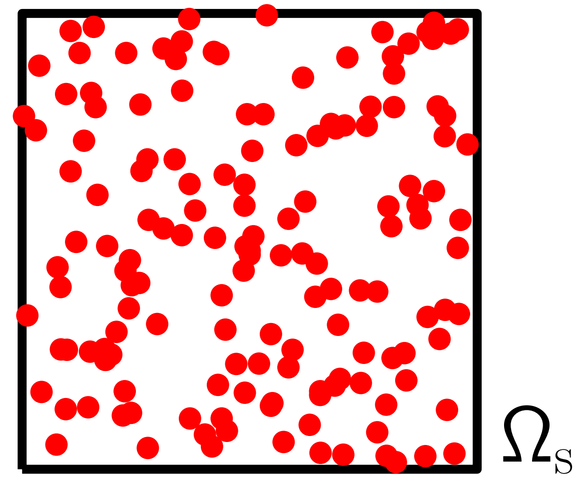
with entries

$$\mathbf{A}(i,j) = \log|\boldsymbol{x}_i - \boldsymbol{y}_j|.$$

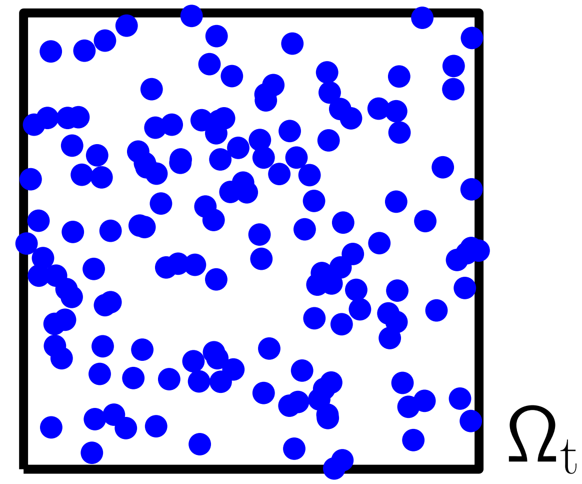
We seek to evaluate

$$\mathbf{q} \mapsto \mathbf{f} = \mathbf{A}\mathbf{q}.$$

Source locations $\{\boldsymbol{y}_j\}_{j=1}^{n}$



Target locations $\{\boldsymbol{x}_i\}_{i=1}^{m}$



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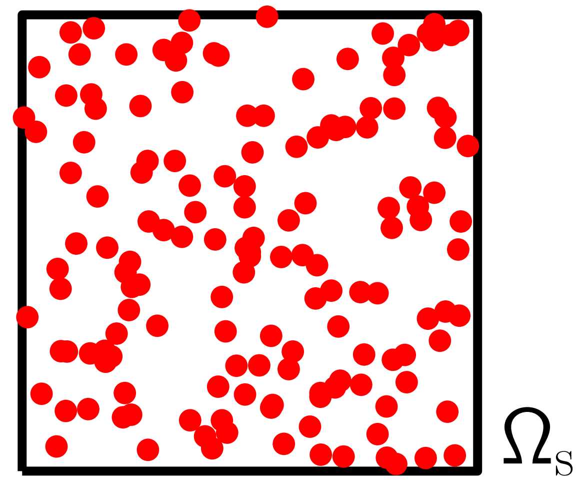

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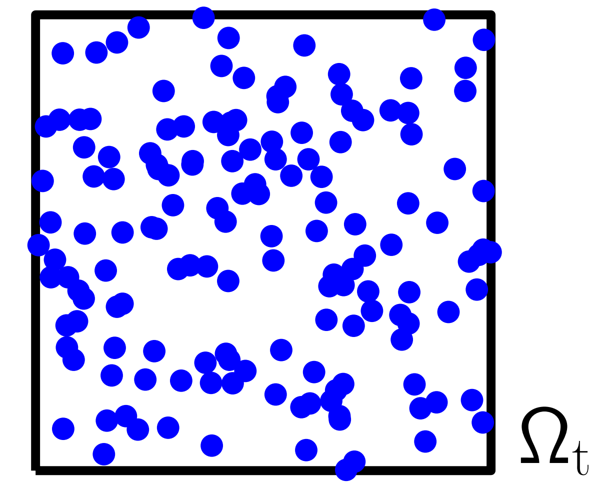


Multipole Expansion: We showed that we can separate variables in the kernel,

$$\log|\boldsymbol{x} - \boldsymbol{y}| = \sum_{p=0}^{\infty} B_p(\boldsymbol{x})\, C_p(\boldsymbol{y}).$$

Using polar coordinates,

$$\boldsymbol{x} - \boldsymbol{c}_{\mathrm{s}} = r\, e^{i\theta}, \qquad \text{and} \qquad \boldsymbol{y} - \boldsymbol{c}_{\mathrm{s}} = r'\, e^{i\theta'},$$

the functions B_p and C_p can (for instance) be

$$\begin{aligned} B_0(\boldsymbol{x}) &= \log r, & C_0(\boldsymbol{y}) &= 1 \\ B_{2p-1}(\boldsymbol{x}) &= -\frac{\sin(p\theta)}{p\, r^p} & C_{2p-1}(\boldsymbol{y}) &= (r')^p \sin(p\theta'), \\ B_{2p}(\boldsymbol{x}) &= -\frac{\cos(p\theta)}{p\, r^p} & C_{2p-1}(\boldsymbol{y}) &= (r')^p \cos(p\theta'). \end{aligned}$$

Upon truncation, we have $\left|\log|\boldsymbol{x} - \boldsymbol{y}| - \sum_{p=0}^{k} B_p(\boldsymbol{x})\, C_p(\boldsymbol{y})\right| \lessapprox (\sqrt{2}/3)^{k/2}$.

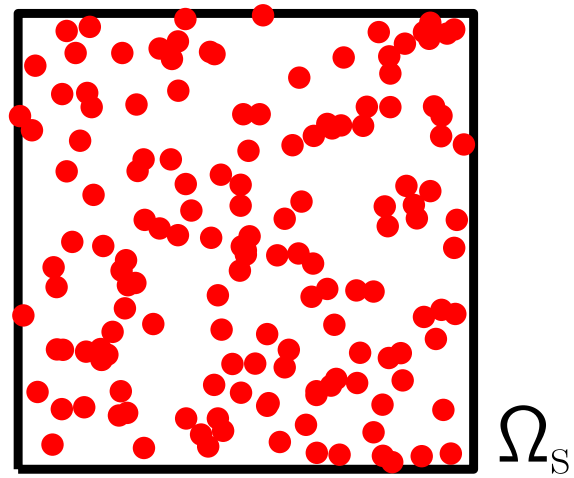
A is the $m \times n$ matrix with entries

$$\mathbf{A}(i,j) = \log|\boldsymbol{x}_i - \boldsymbol{y}_j|.$$

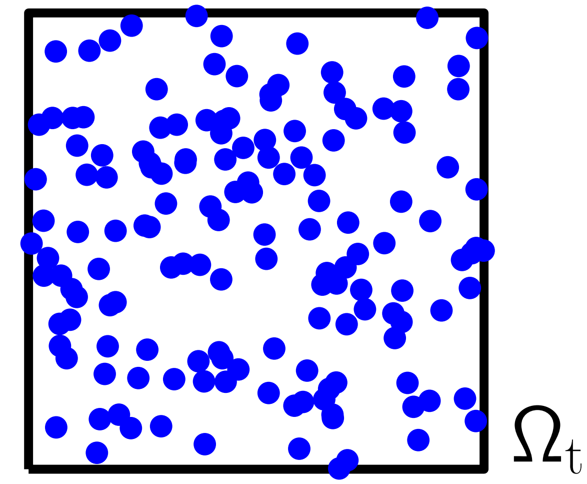
We seek to evaluate

$$\mathbf{q} \mapsto \mathbf{f} = \mathbf{Aq}.$$

Source locations $\{\boldsymbol{y}_j\}_{j=1}^{n}$



Target locations $\{\boldsymbol{x}_i\}_{i=1}^{m}$



Multipole Expansion: The precise form of the factors is not directly relevant for the discussion at hand, so to keep the notation uncluttered, let us simply write the approximation as

$$\log|\boldsymbol{x} - \boldsymbol{y}| \approx \sum_{p=1}^{k} B_p(\boldsymbol{x})\, C_p(\boldsymbol{y}).$$

Note that we truncated the expansion after k terms, incurring an error $\approx (\sqrt{2}/3)^{k/2}$.

(We changed the summation index to start at 1, too.)

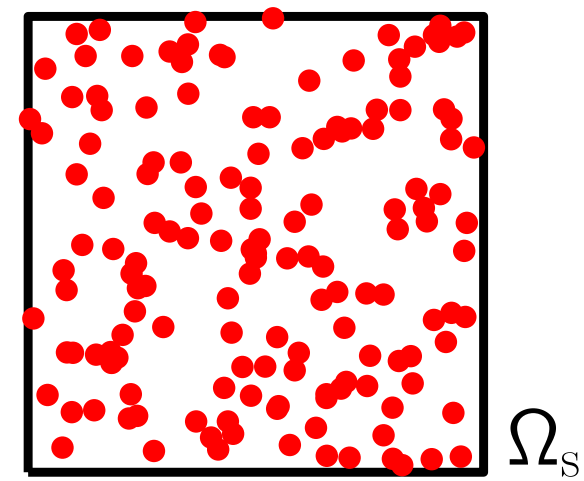
A is the $m \times n$ matrix with entries

$$\mathbf{A}(i,j) = \log|\boldsymbol{x}_i - \boldsymbol{y}_j|.$$

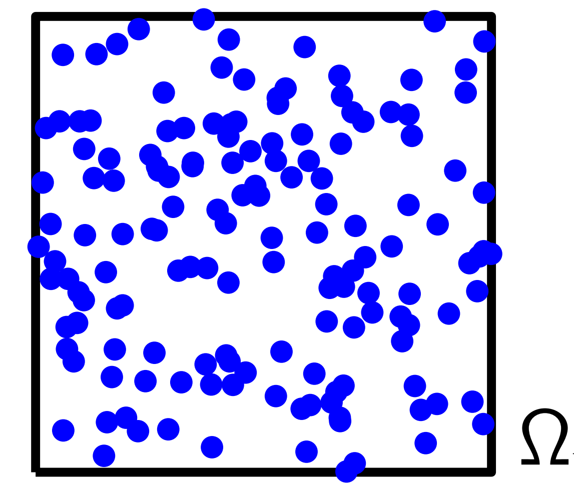
We seek to evaluate

$$\mathbf{q} \mapsto \mathbf{f} = \mathbf{A}\mathbf{q}.$$

Source locations $\{\boldsymbol{y}_j\}_{j=1}^{n}$



Target locations $\{\boldsymbol{x}_i\}_{i=1}^{m}$



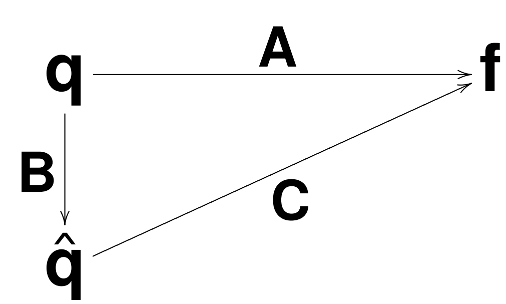
Multipole Expansion: Recall the k term multipole expansion:

$$\log|\boldsymbol{x} - \boldsymbol{y}| \approx \sum_{p=1}^{k} B_p(\boldsymbol{x})\, C_p(\boldsymbol{y}). \tag{1}$$

An approximation (1) is called a *separation of variables,* and directly leads to a low-rank factorization

$$\begin{array}{ccccc} \mathbf{A} & \approx & \mathbf{B} & \mathbf{C}. \\ m \times n & & m \times k & k \times n \end{array}$$

where **B** has entries $\mathbf{B}(i,p) = B_p(\boldsymbol{x}_i)$ and **C** has entries $\mathbf{C}(p,j) = C_p(\boldsymbol{y}_j)$.



Reduction in cost: *From mn flops to $2k(m+n)$ flops,* where $k \sim \log(1/\varepsilon)$.

Suppose **A** is a given $m \times n$ matrix.

Question: What is the theoretically "best" factorization of **A** for any given ε?

Answer: Consider the *singular value decomposition (SVD)* of **A**:

$$\begin{array}{cccc} \mathbf{A} & \approx \quad \mathbf{U} & \mathbf{D} & \mathbf{V}^*. \\ m \times n & m \times r & r \times r & r \times n \end{array}$$

where $r = \min(m, n)$ and where

$\mathbf{U} = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_r]$	is a matrix holding the "left singular vectors" $\mathbf{u}_i$,
$\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_r]$	is a matrix holding the "right singular vectors" $\mathbf{v}_i$,
$\mathbf{D} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$	is a diagonal matrix holding the "singular values" σ_i.

Let $||\cdot||$ denote a matrix norm and let e_k denote the minimal error in a rank-k factorization

$$e_k = \min\{||\mathbf{A} - \mathbf{A}_k|| : \ \mathbf{A}_k \text{ has rank } k\}.$$

Theorem (Eckart-Young): The minimal error is

$$e_k = \sigma_{k+1}, \qquad \text{when the spectral norm is used}$$

$$e_k = \sqrt{\sigma_{k+1}^2 + \sigma_{k+2}^2 + \cdots + \sigma_r^2}, \qquad \text{when the Frobenius norm is used}$$

and the minimal error is attained for the SVD truncated to the first k terms

$$e_k = ||\mathbf{A} - \sum_{j=1}^{k} \mathbf{u}_j\, \sigma_j\, \mathbf{v}_j^*|| = ||\mathbf{A} - \mathbf{U}_k\, \mathbf{D}_k\, \mathbf{V}_k^*||.$$

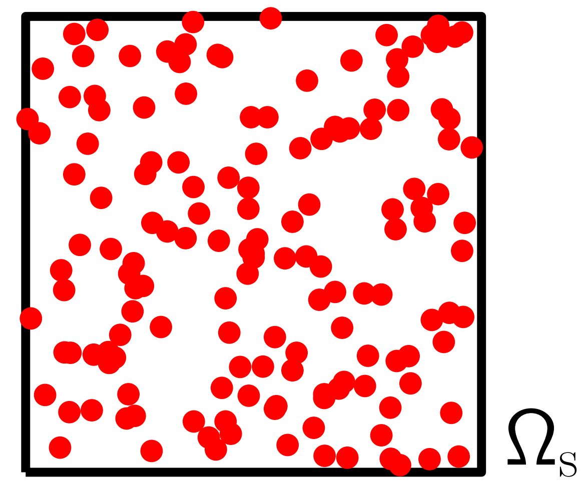
A is the $m \times n$ matrix with entries

$$\mathbf{A}(i,j) = \log|\boldsymbol{x}_i - \boldsymbol{y}_j|.$$

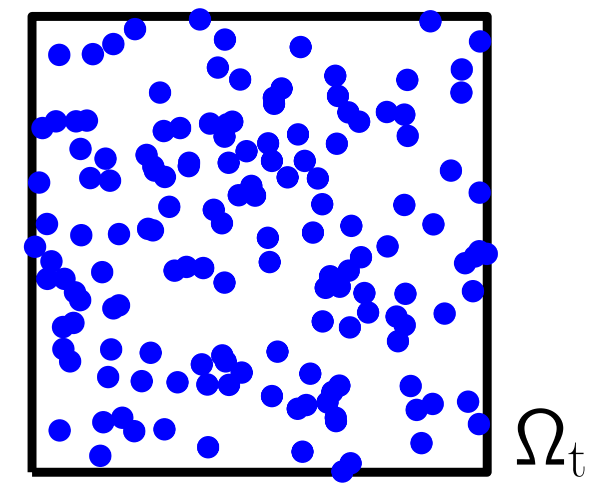
We seek to evaluate

$$\mathbf{q} \mapsto \mathbf{f} = \mathbf{A}\mathbf{q}.$$

Source locations $\{\boldsymbol{y}_j\}_{j=1}^{n}$



Target locations $\{\boldsymbol{x}_i\}_{i=1}^{m}$



Optimal factorization — SVD: Compute the SVD of **A**, and pick k such that $\sigma_{k+1} \leq \varepsilon$. Set $\mathbf{B} = \mathbf{U}_k$ and $\mathbf{C} = \mathbf{D}_k\,\mathbf{V}_k^*$. Then

$$\underset{m\times n}{\mathbf{A}} \approx \underset{m\times k}{\mathbf{B}}\ \underset{k\times n}{\mathbf{C}}$$

is the theoretically most economical factorization of **A**.

However, the SVD is not quite ideal:

- All factors are determined numerically — expensive!
- The factors **B** and **C** depend on the precise geometry.

☞ You have to custom-build all translation operators.

We will next describe a factorization that is almost optimal, and is also easy and economical to work with.

The Interpolative Decomposition (ID):

Let **A** be an $m \times n$ matrix of (precise) rank k. Then **A** admits a factorization

$$\underset{m \times n}{\mathbf{A}} = \underset{m \times k}{\mathbf{A}^{(\mathrm{skel})}} \; \underset{k \times n}{\mathbf{V}^*},$$

where

1. $\mathbf{A}^{(\mathrm{skel})} = \mathbf{A}(:, \tilde{I})$ consists of *k columns of **A**.*
2. **V** contains a $k \times k$ identity matrix.
3. No entry of **V** has magnitude greater than 1 (so **V** is reasonably well-conditioned).

How do you construct an ID in practice?

- Computing an ID that satisfies (3) is (in general) very hard.
- If we relax condition (3) slightly, and require only that, say, $\max_{ij} |\mathbf{V}(i,j)| \leq 1.1$, then it can be done efficiently [1996, Gu & Eisenstat].
- In practice, simply performing Gram-Schmidt on the columns works great.
  After k steps of column pivoted QR, we have

$$\mathbf{A}(:, I) = \mathbf{Q}\,[\mathbf{R}_{11} \quad \mathbf{R}_{12}] = \underbrace{\mathbf{Q}\mathbf{R}_{11}}_{=\mathbf{A}^{(\mathrm{skel})}} \underbrace{[\mathbf{I} \quad \mathbf{R}_{11}^{-1}\mathbf{R}_{12}]}_{=\mathbf{V}^*}.$$

- If **A** does not have exact rank k, *but its singular values decay rapidly,* then the ID resulting from Gram-Schmidt satisfies $||\mathbf{A} - \mathbf{A}^{(\mathrm{skel})}\mathbf{V}^*|| \approx \sigma_{k+1}$.

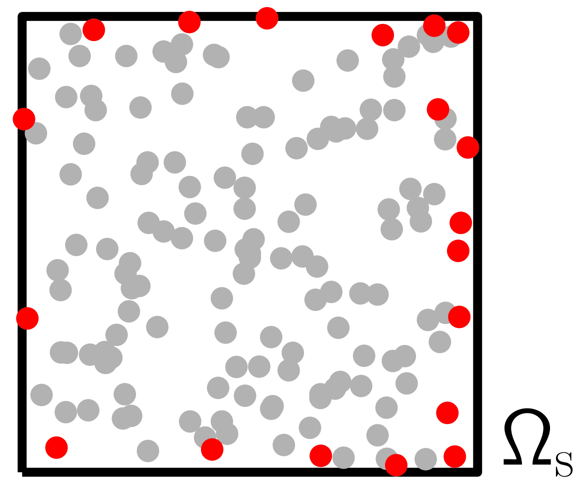
$\mathbf{A}$ is the $m \times n$ matrix with entries

$$\mathbf{A}(i,j) = \log|\boldsymbol{x}_i - \boldsymbol{y}_j|.$$

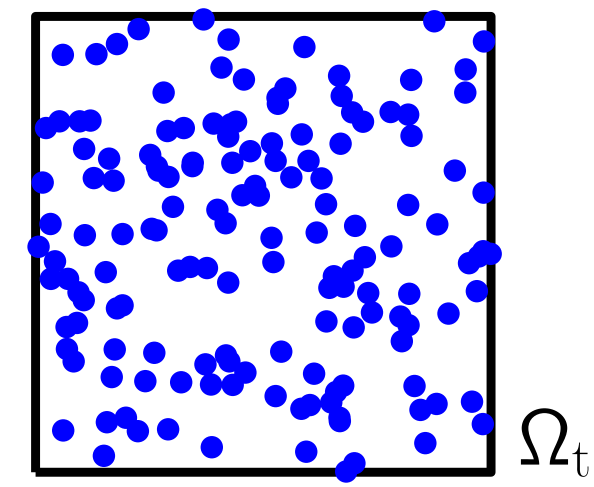
We seek to evaluate

$$\mathbf{q} \mapsto \mathbf{f} = \mathbf{A}\mathbf{q}.$$

Source locations $\{\boldsymbol{y}_j\}_{j=1}^{n}$



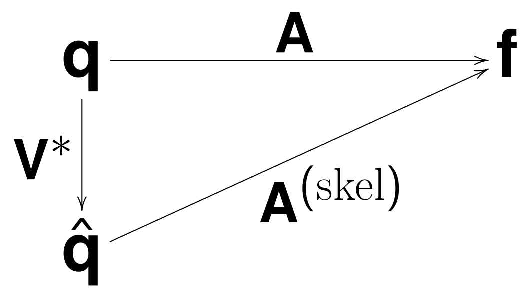
Target locations $\{\boldsymbol{x}_i\}_{i=1}^{m}$



Interpolative decomposition (ID): Performing G-S on the columns of $\mathbf{A}$, we obtain

$$\begin{array}{ccc} \mathbf{A} & \approx \mathbf{A}^{(\rm skel)} & \mathbf{V}^* \\ m \times n & m \times k & k \times n \end{array}$$

where $\mathbf{A}^{(\rm skel)} = \mathbf{A}(:,\tilde{I})$ consists of k columns of $\mathbf{A}$.



The nodes marked in red above are the nodes marked by the index vector $\tilde{I}$.

The interaction of $\Omega_{\rm s}$ with the outside is through the original kernel function.

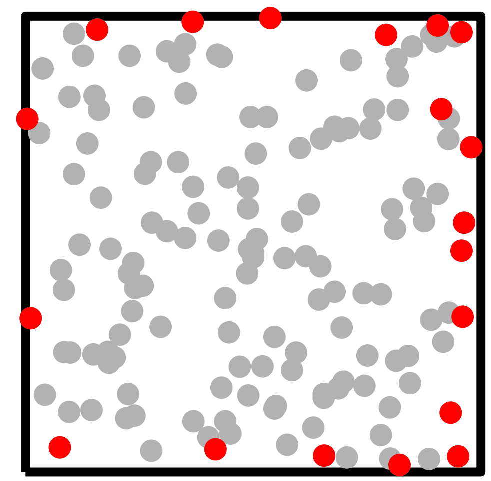
A is the $m \times n$ matrix with entries

$$\mathbf{A}(i,j) = \log|\boldsymbol{x}_i - \boldsymbol{y}_j|.$$

We seek to evaluate

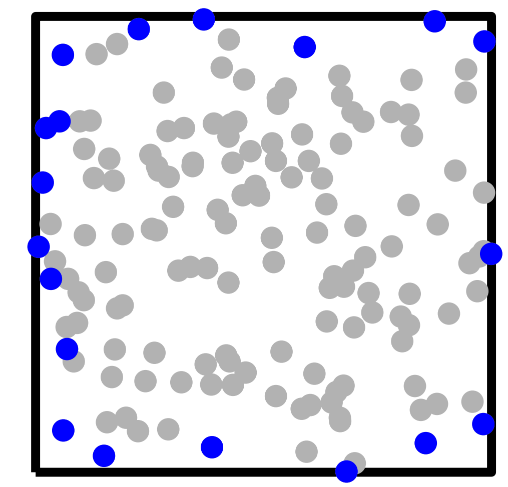
$$\mathbf{q} \mapsto \mathbf{f} = \mathbf{A}\mathbf{q}.$$

Source locations $\{\boldsymbol{y}_j\}_{j=1}^{n}$



$\Omega_{\rm s}$

Target locations $\{\boldsymbol{x}_i\}_{i=1}^{m}$

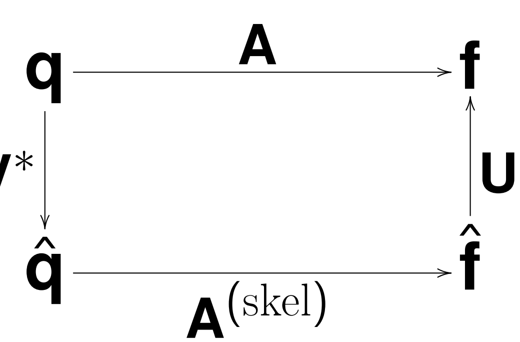


$\Omega_{\rm t}$

Interpolative decomposition (ID): Let's do G-S on the *rows* of **A** as well

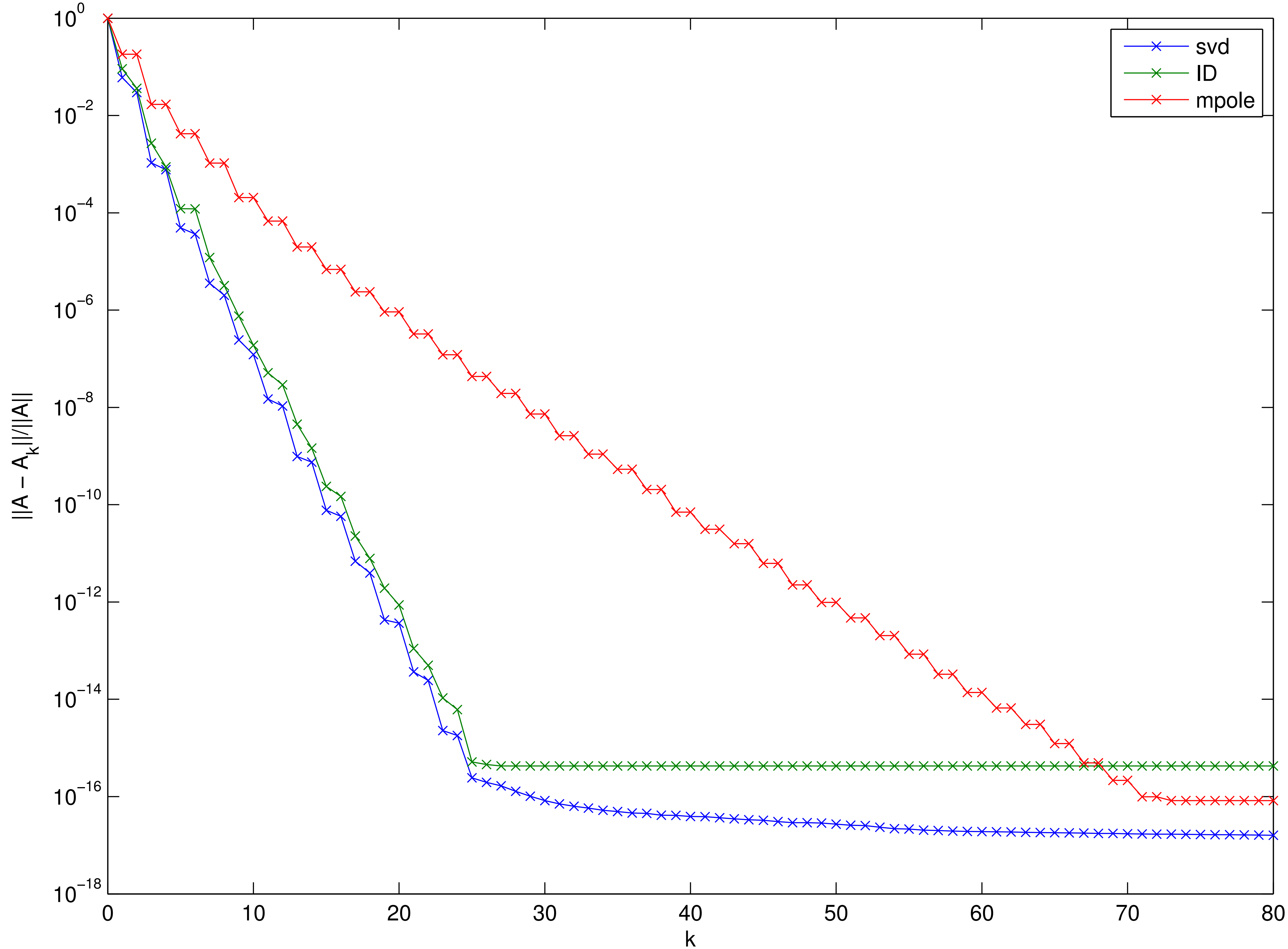
$$\begin{array}{ccccc} \mathbf{A} & \approx & \mathbf{U} & \mathbf{A}^{(\rm skel)} & \mathbf{V}^* \\ m \times n & & m \times k & k \times k & k \times n \end{array}$$

where $\mathbf{A}^{(\rm skel)} = \mathbf{A}(\tilde{I}_{\rm t}, \tilde{I}_{\rm s})$ is a $k \times k$ sub-matrix of **A**.



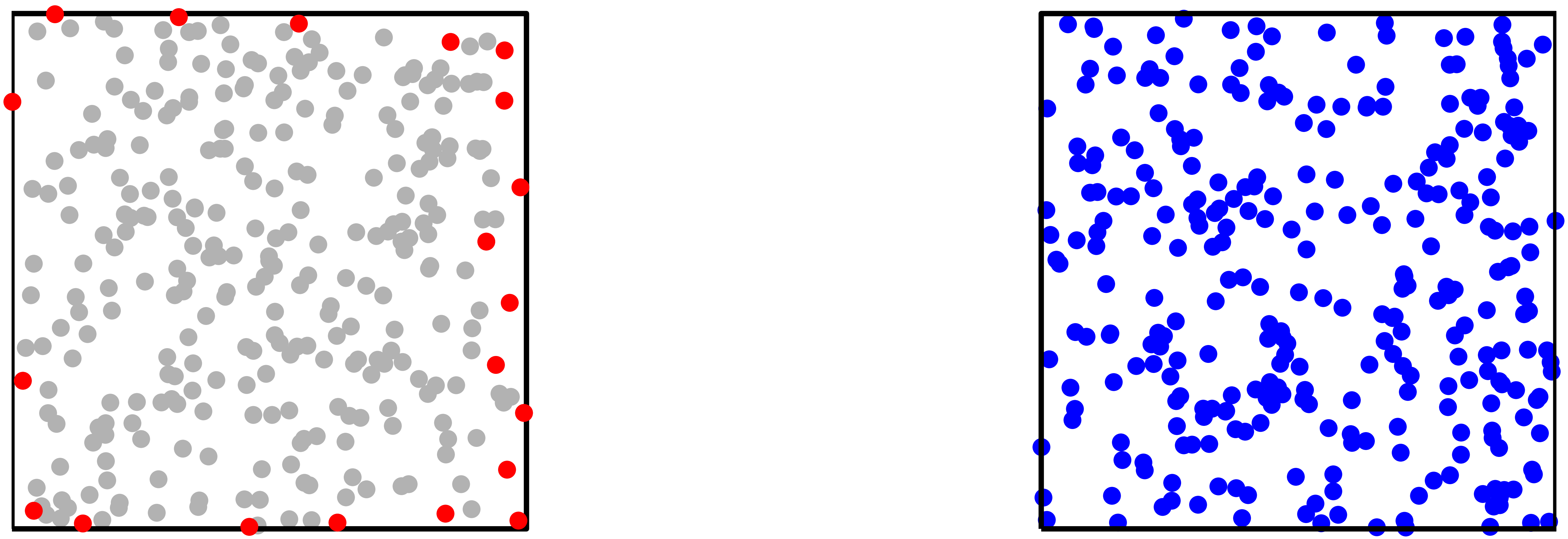
Approximation errors as a function of the rank k.

Interaction potential is Laplace, $\mathbf{A}(i,j) = \log|\boldsymbol{x}_i - \boldsymbol{y}_j|$.



The 20 skeleton points required for (relative) accuracy $\varepsilon = 10^{-12}$.

Interaction potential is Laplace, $\mathbf{A}(i,j) = \log|\boldsymbol{x}_i - \boldsymbol{y}_j|$.

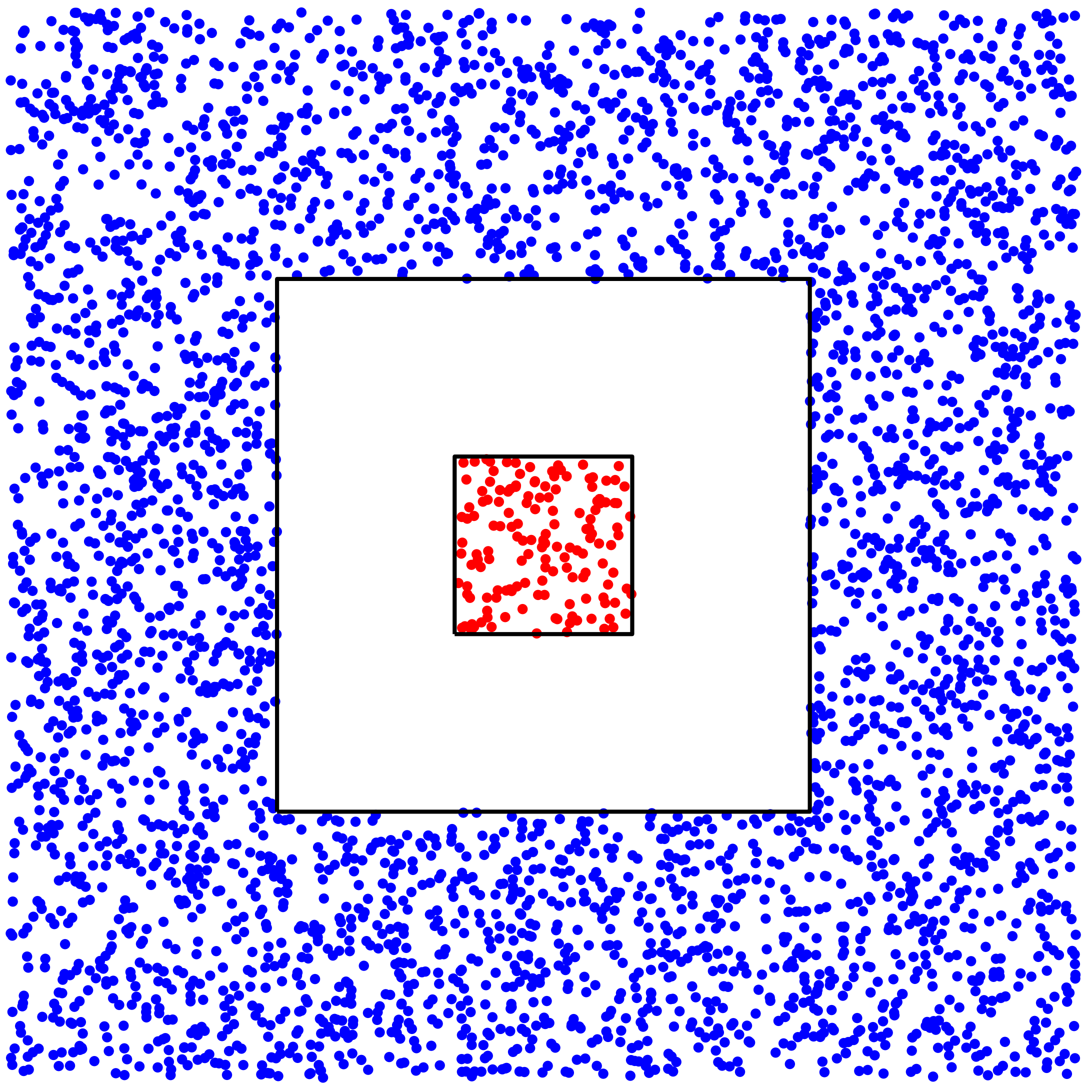


Conclusions from experiments:

- The SVD and ID are comparable in effectiveness.
  (In our case! When the singular values decay slowly, this is not true.)
- The multipole expansion requires more terms.

But, the comparison is not quite fair — the multipole expansion is valid for *any* source point that is well-separated.

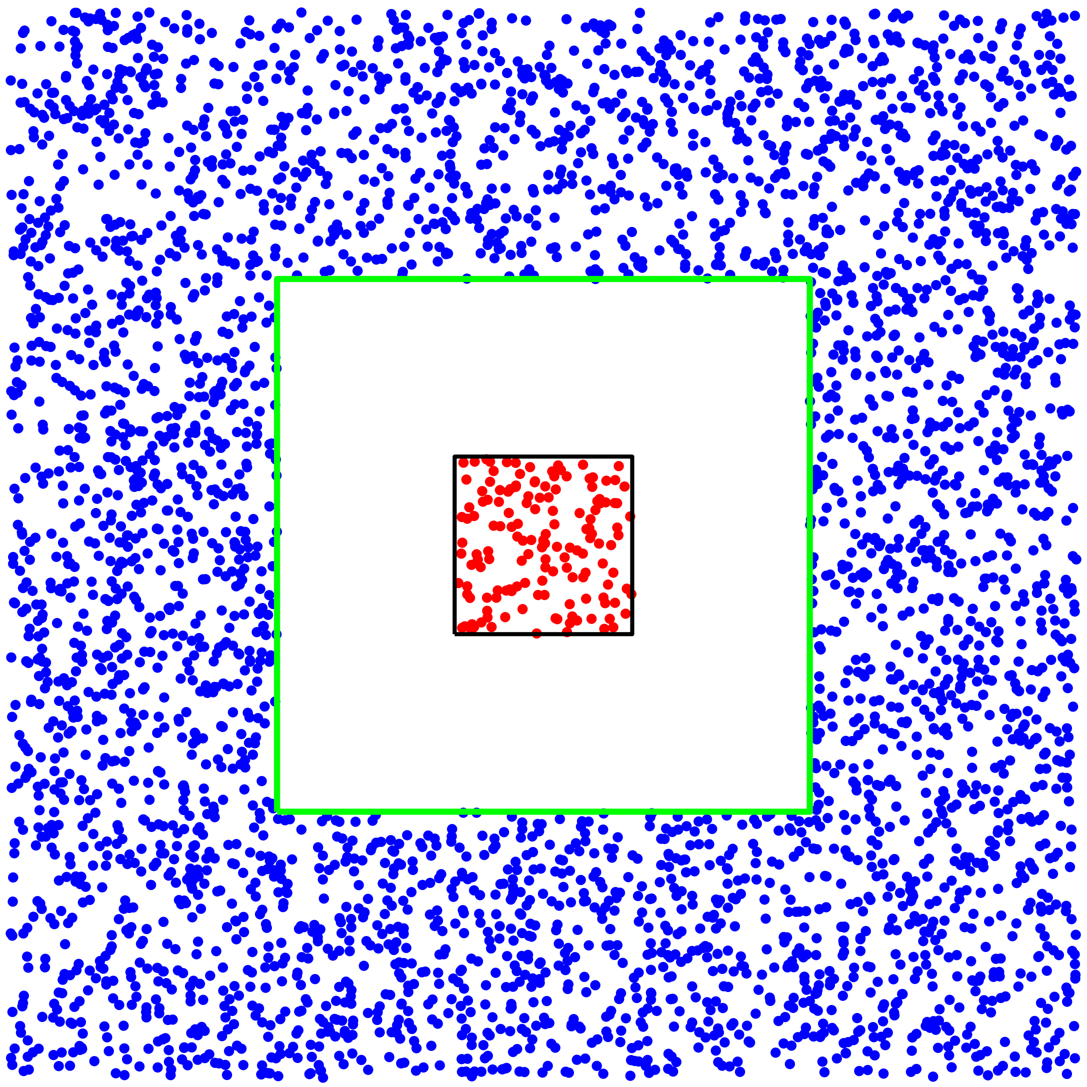
Question: Can we find skeleton points that "work" for any well-separated target point?



First observe that we do not need to consider "every" potential target point.

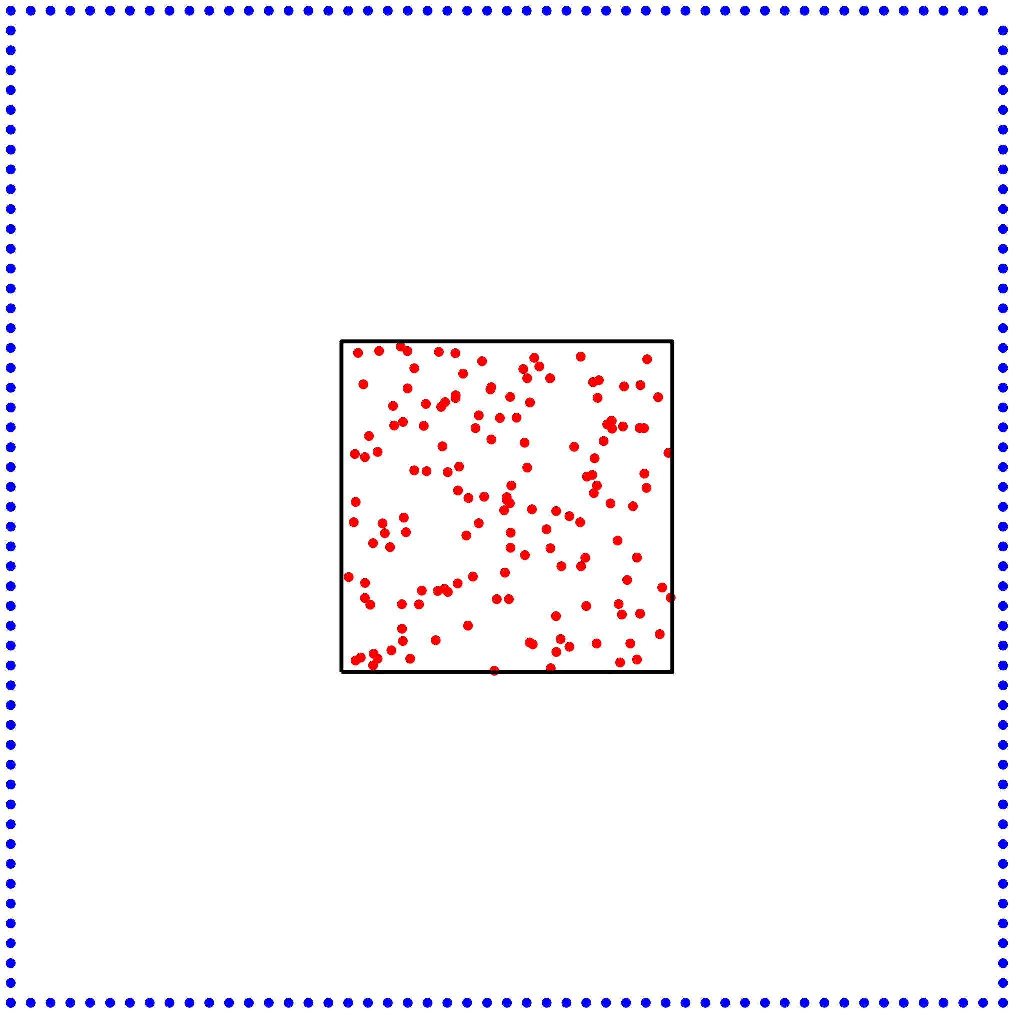
Let u denote the potential caused by the source points: $u(\boldsymbol{x}) = \sum_{j=1}^{n} q_j \log|\boldsymbol{x} - \boldsymbol{y}_j|$.

Now suppose that we can accurately reconstruct u on the green square shown:



Observe that u is harmonic (i.e. $-\Delta u = 0$) outside the green square.

Since the Laplace problem has a unique solution, we know that if we correctly reproduce u on the green square, then it is correctly reproduced *everywhere* outside the square.

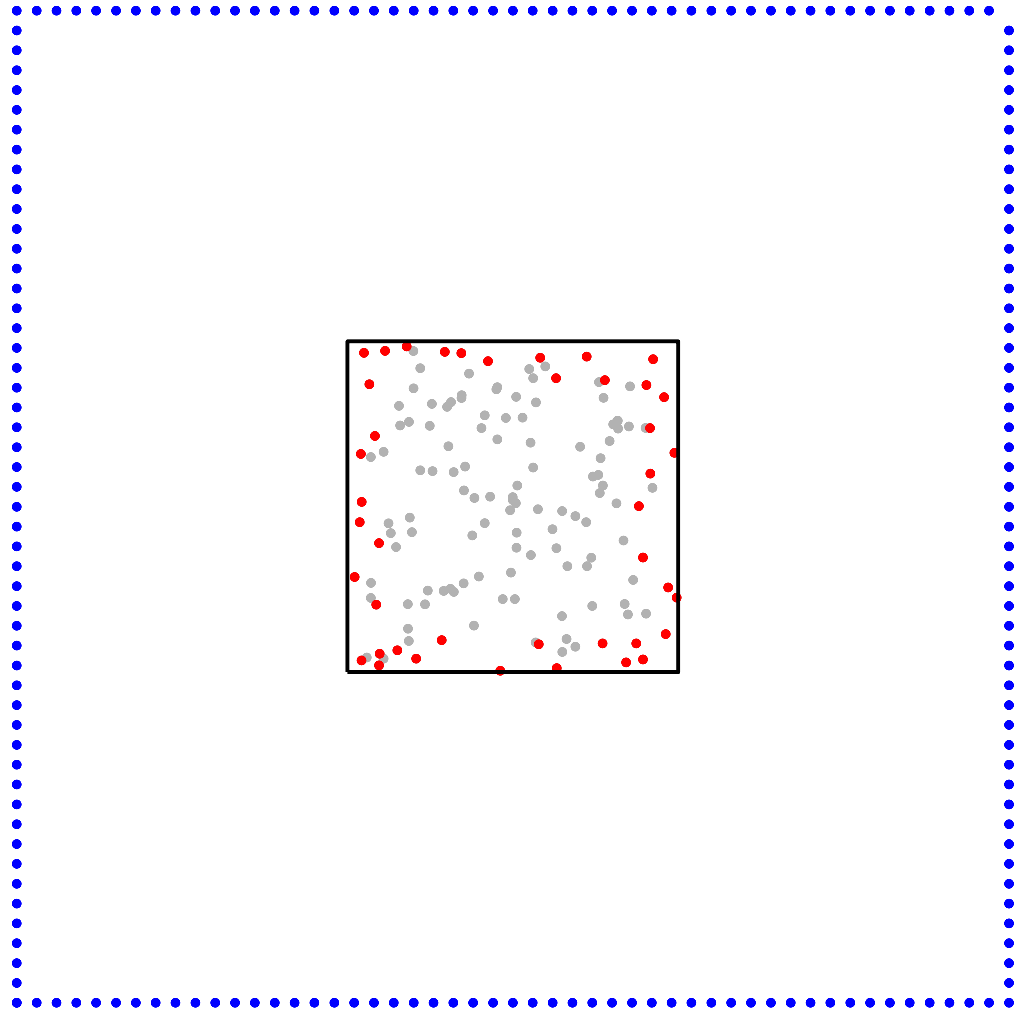


Let $\{\boldsymbol{y}_j\}_{j=1}^{n}$ be sources in the small red box.

Let $\{\boldsymbol{x}_i\}_{i=1}^{m}$ be targets on the large blue box.

Let $\mathbf{A}$ be the matrix with elements $\mathbf{A}(i,j) = \log|\boldsymbol{x}_i - \boldsymbol{y}_j|$.

Perform Gram-Schmidt on the columns of $\mathbf{A}$,



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Let $\mathbf{A}$ be the matrix with elements $\mathbf{A}(i,j) = \log|\boldsymbol{x}_i - \boldsymbol{y}_j|$.

Perform Gram-Schmidt on the columns of $\mathbf{A}$,

$$\begin{array}{ccccc} \mathbf{A} & \approx & \mathbf{A}^{(\mathrm{skel})} & \mathbf{V}^* \\ m \times n & & m \times k & k \times n \end{array}$$

We *know* that (to within precision ε), this skeleton is valid at any well-separated point.

For $\varepsilon = 10^{-12}$, we now have $k = 45$. It as $k = 20$ for the two-box geometry.

One concern remains: So far, we've looked at a given distribution of source locations. The skeleton points chosen are not "universal".

To address this issue, we will henceforth investigate the *continuum operator* A:

$$f(\boldsymbol{x}) = [Aq](\boldsymbol{x}) = \int_{\Omega_{\rm s}} \log|\boldsymbol{x}-\boldsymbol{y}|\, q(\boldsymbol{y})\, d\boldsymbol{y}, \qquad \boldsymbol{x} \in \Omega_{\rm t}$$

which maps a source distribution q in a source domain $\Omega_{\rm s}$ to a potential f in a target domain $\Omega_{\rm t}$.

Let $\{\boldsymbol{x}_i, v_i\}_{i=1}^{m}$ be a quadrature for the target domain, and
let $\{\boldsymbol{y}_j, w_j\}_{j=1}^{n}$ be a quadrature for the source domain.

Let the vector $\mathbf{f}$ have entries $\mathbf{f}(i) = \sqrt{v_i}\, f(\boldsymbol{x}_i)$ so that $||f||_{L^2(\Omega_{\rm t})} \approx ||\mathbf{f}||_{\ell^2}$.

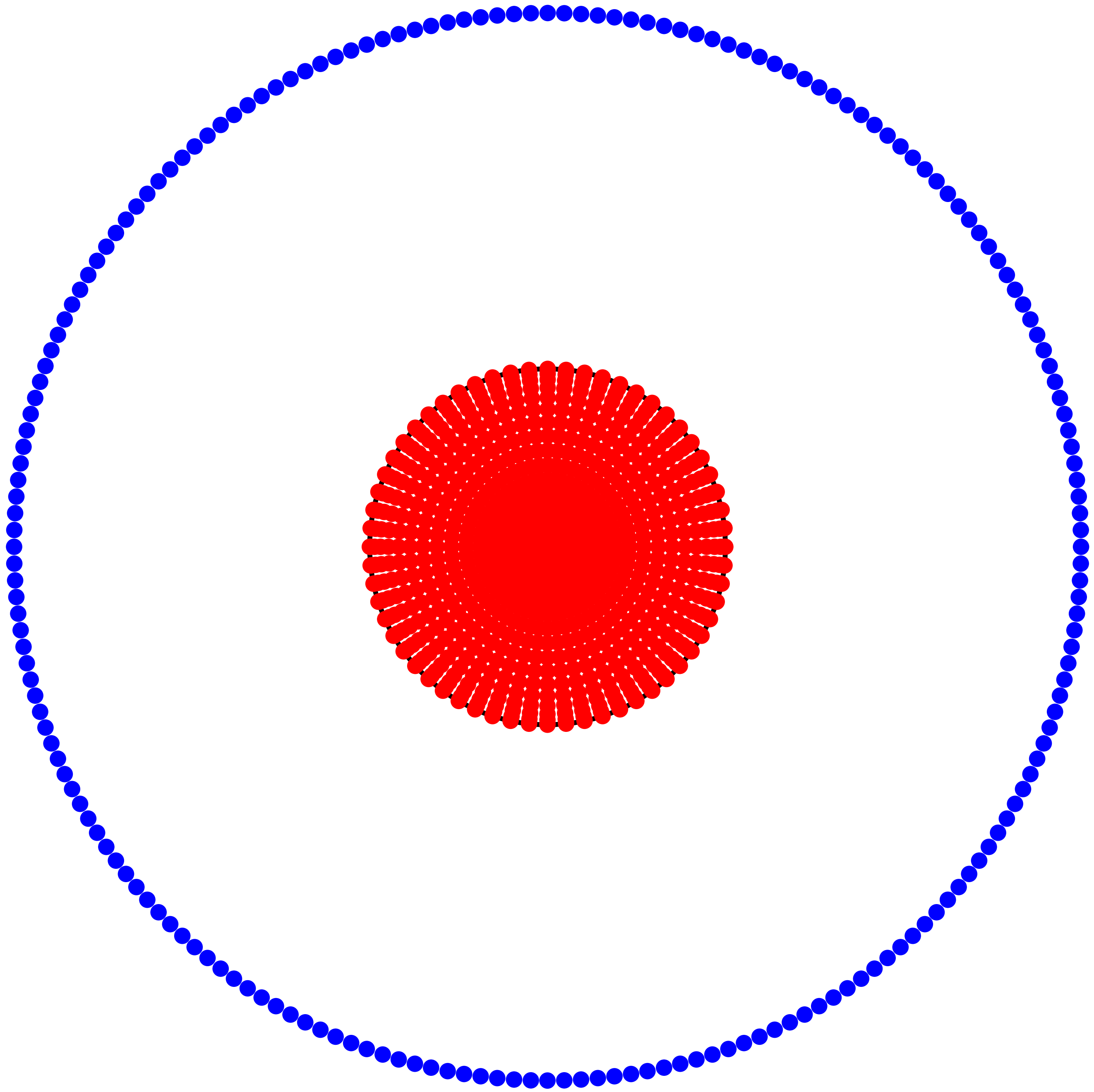
Let the vector $\mathbf{q}$ have entries $\mathbf{q}(j) = \sqrt{w_j}\, q(\boldsymbol{y}_j)$ so that $||q||_{L^2(\Omega_{\rm s})} \approx ||\mathbf{q}||_{\ell^2}$.

Finally, let $\mathbf{A}$ be the $m \times n$ matrix with entries $\mathbf{A}(i,j) = \sqrt{v_i}\, \log|\boldsymbol{x}_i - \boldsymbol{y}_j|\, \sqrt{w_j}$.

Then the singular values/vectors of $\mathbf{A}$ are accurate approximations of the singular values/vectors of A.

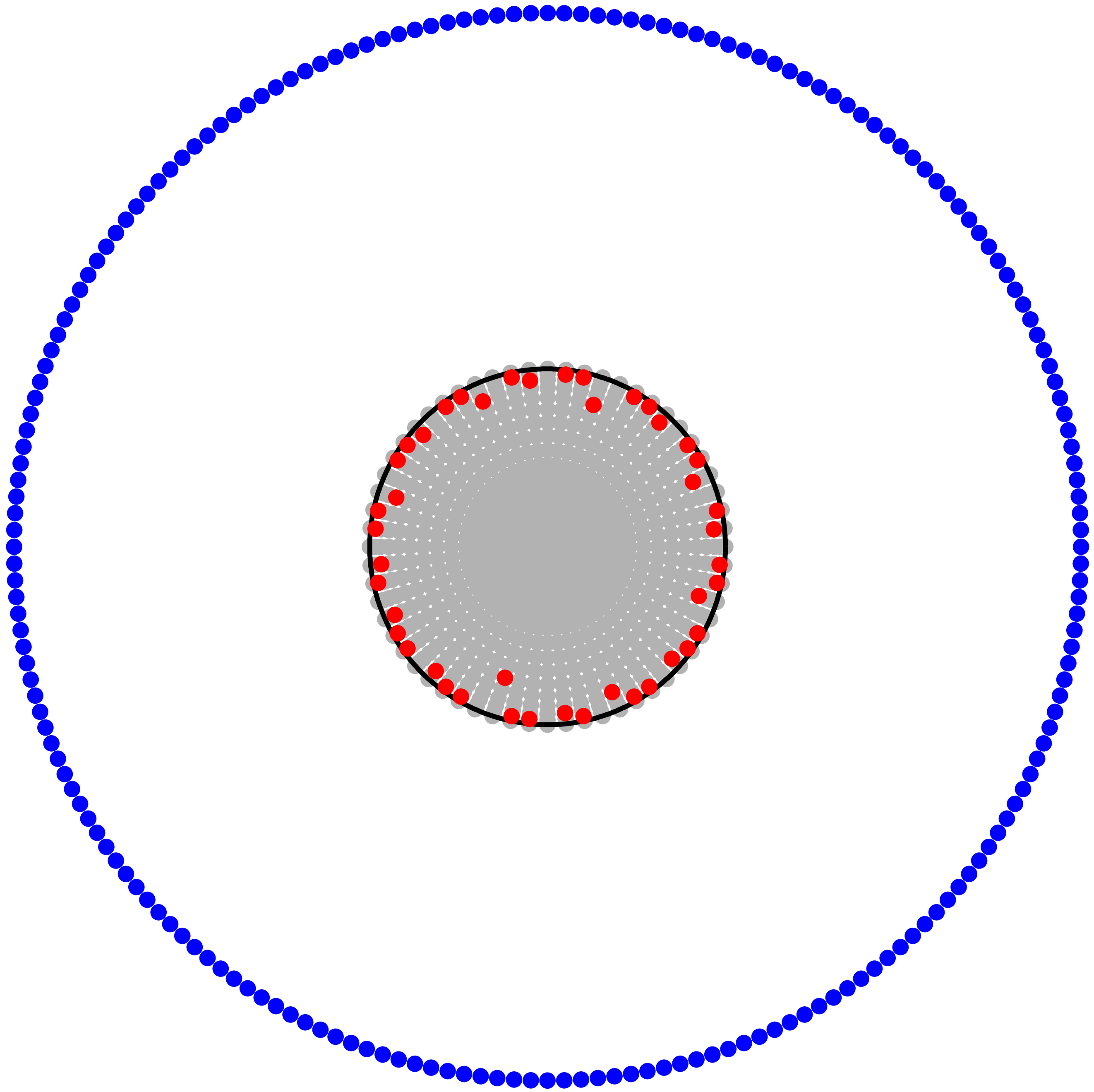
Observe that when $\Omega_{\rm s}$ and $\Omega_{\rm t}$ are not "too close," the kernel $\log|\boldsymbol{x}-\boldsymbol{y}|$ is smooth.

Example: Two concentric circles — ideal for multipole expansion.



Sources in a disc of radius 0.5, targets on a circle of radius 1.5.

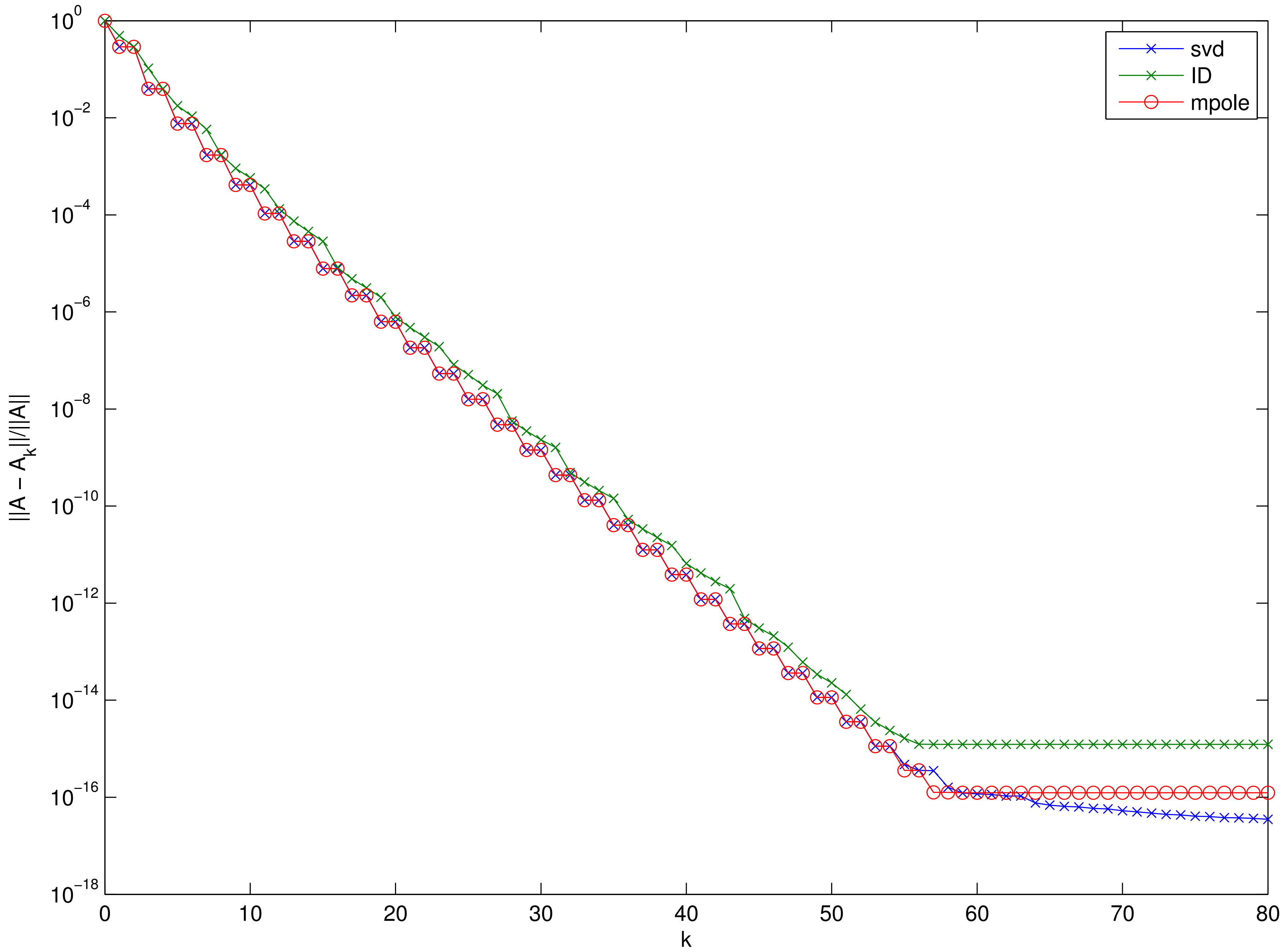
Example: Two concentric circles — ideal for multipole expansion.



Sources in a disc of radius 0.5, targets on a circle of radius 1.5.

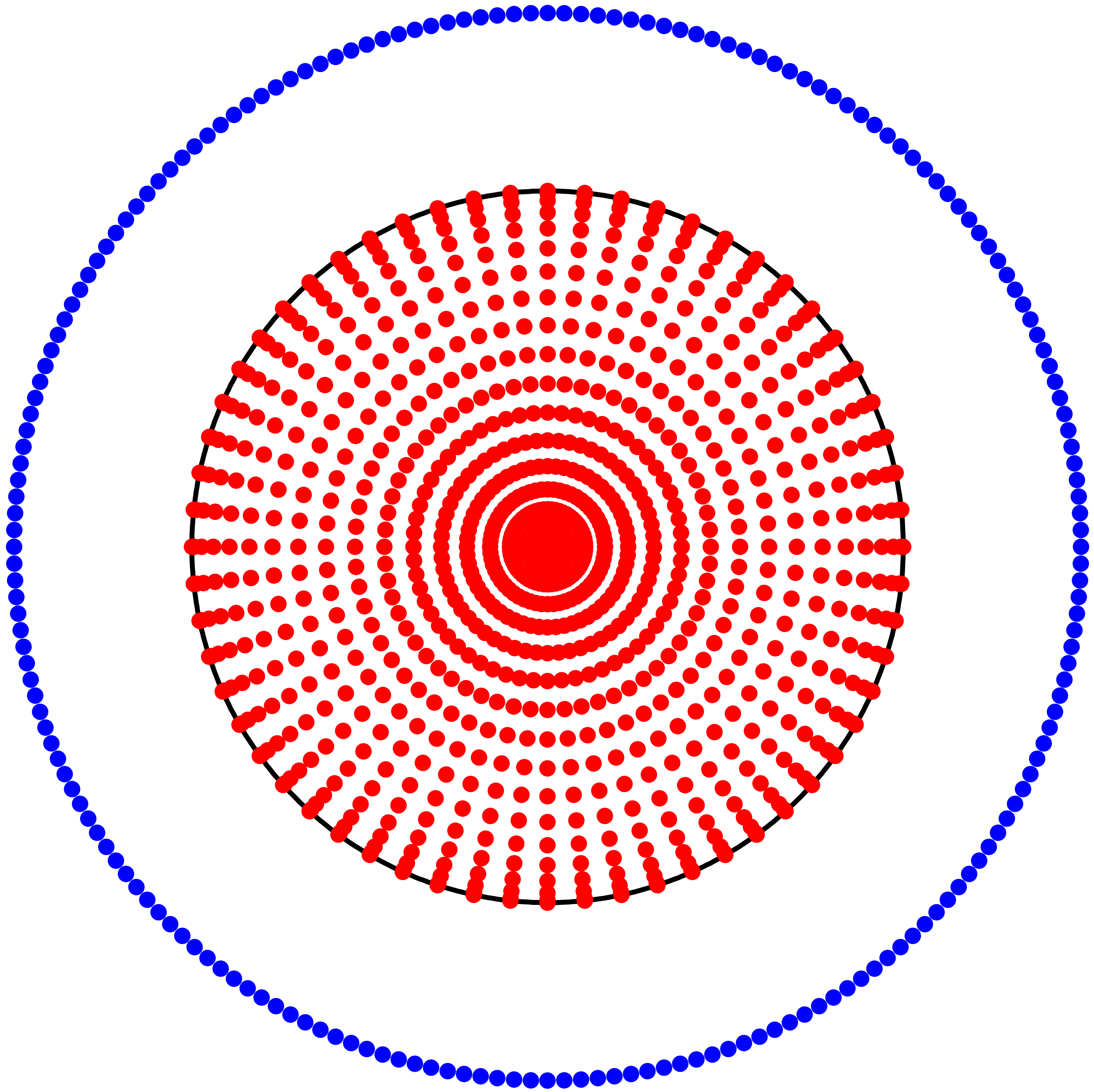
Skeleton to precision $\varepsilon = 10^{-12}$, *which requires* $k = 44$.

Example: Two concentric circles — ideal for multipole expansion.



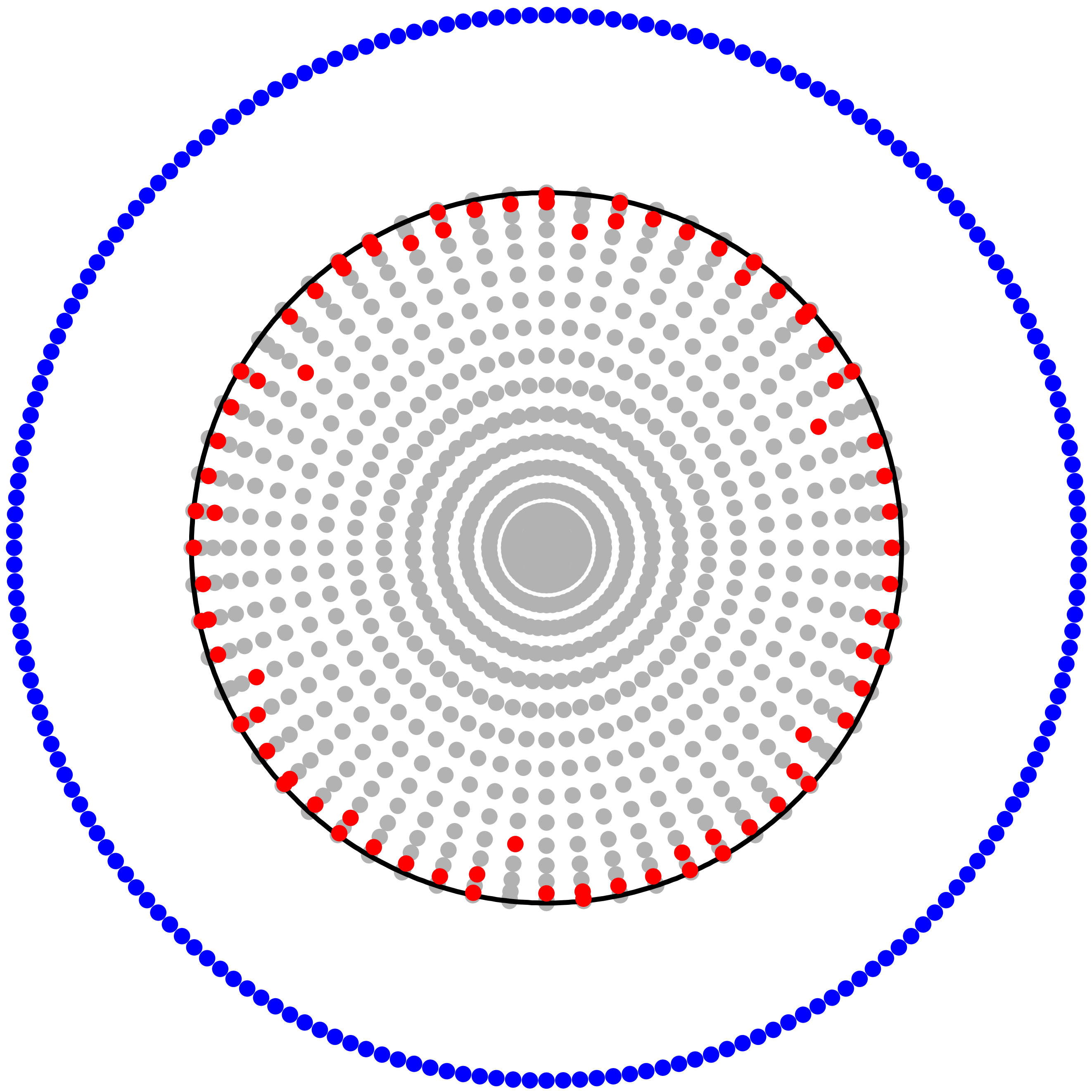
Errors. For this geometry, $E_{\rm mpole} = E_{\rm svd}$ *exactly!*

Example: Two concentric circles — now much tighter.



Sources in a disc of radius 0.5, targets on a circle of radius 0.75.

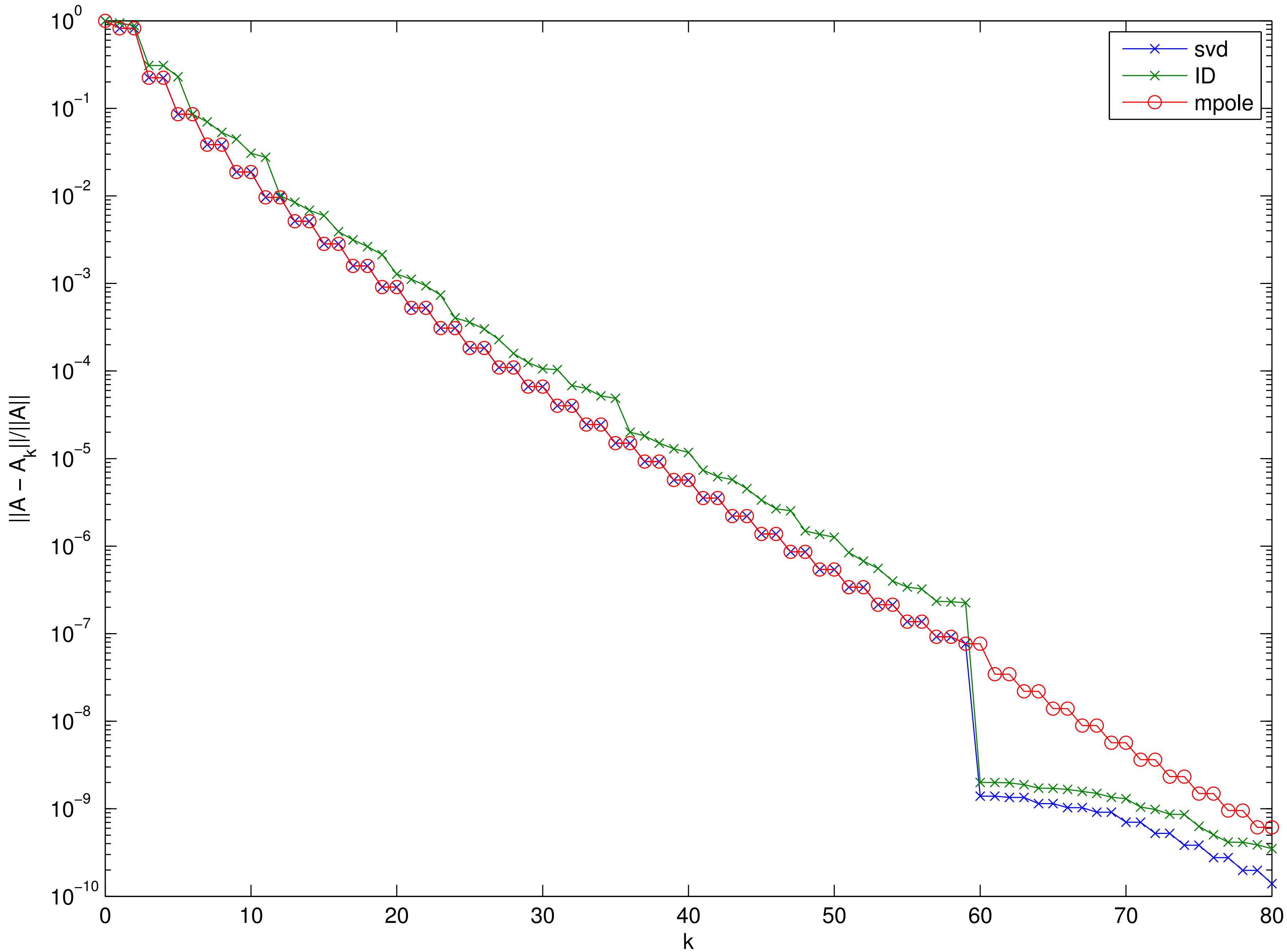
Example: Two concentric circles — now much tighter.



Sources in a disc of radius 0.5, targets on a circle of radius 1.5.

Skeleton to precision $\varepsilon = 10^{-12}$*, which requires* $k = 81$*.*

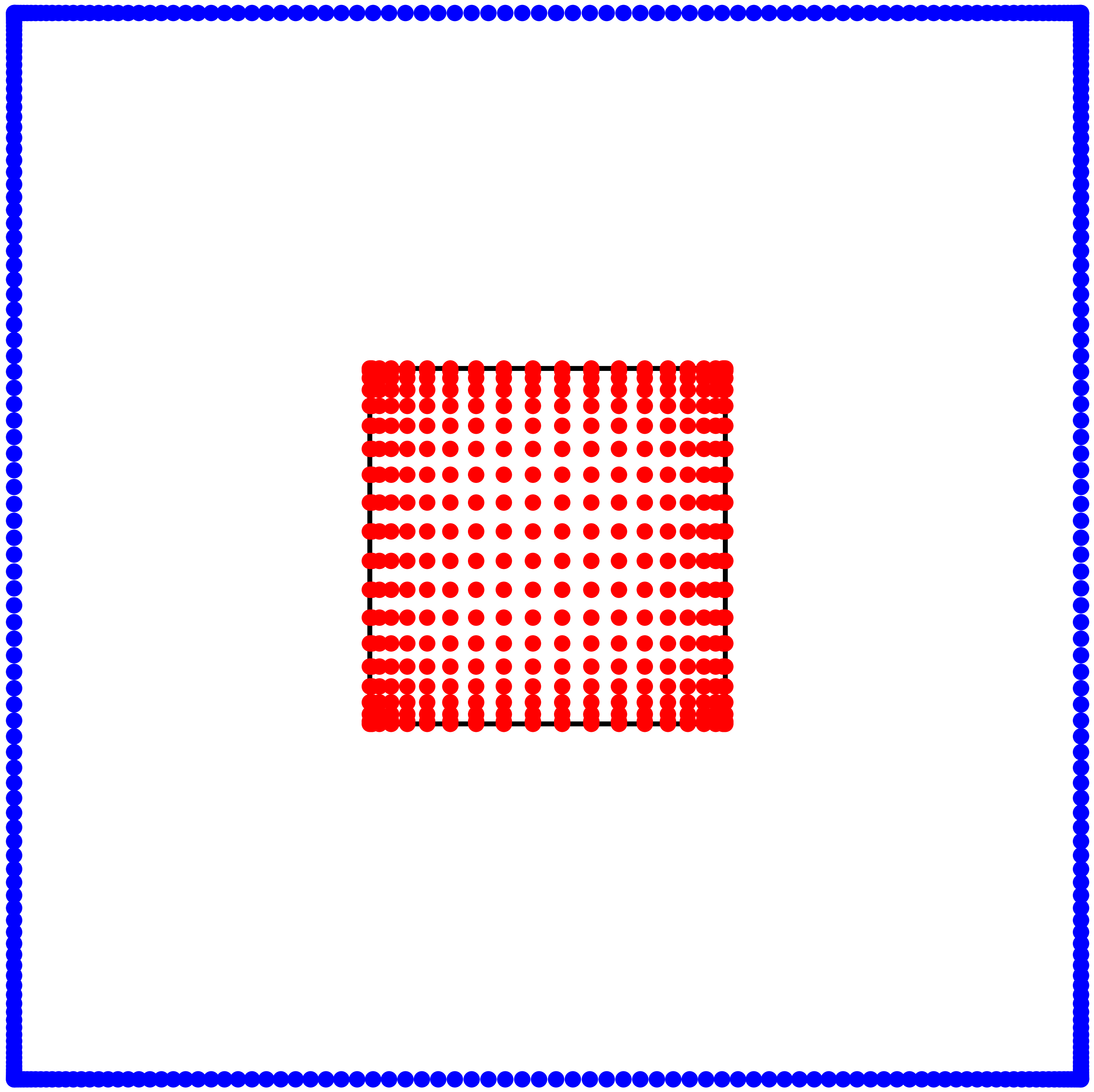
Example: Two concentric circles — now much tighter.



Errors. For this geometry, $E_{\rm mpole} = E_{\rm svd}$ exactly!

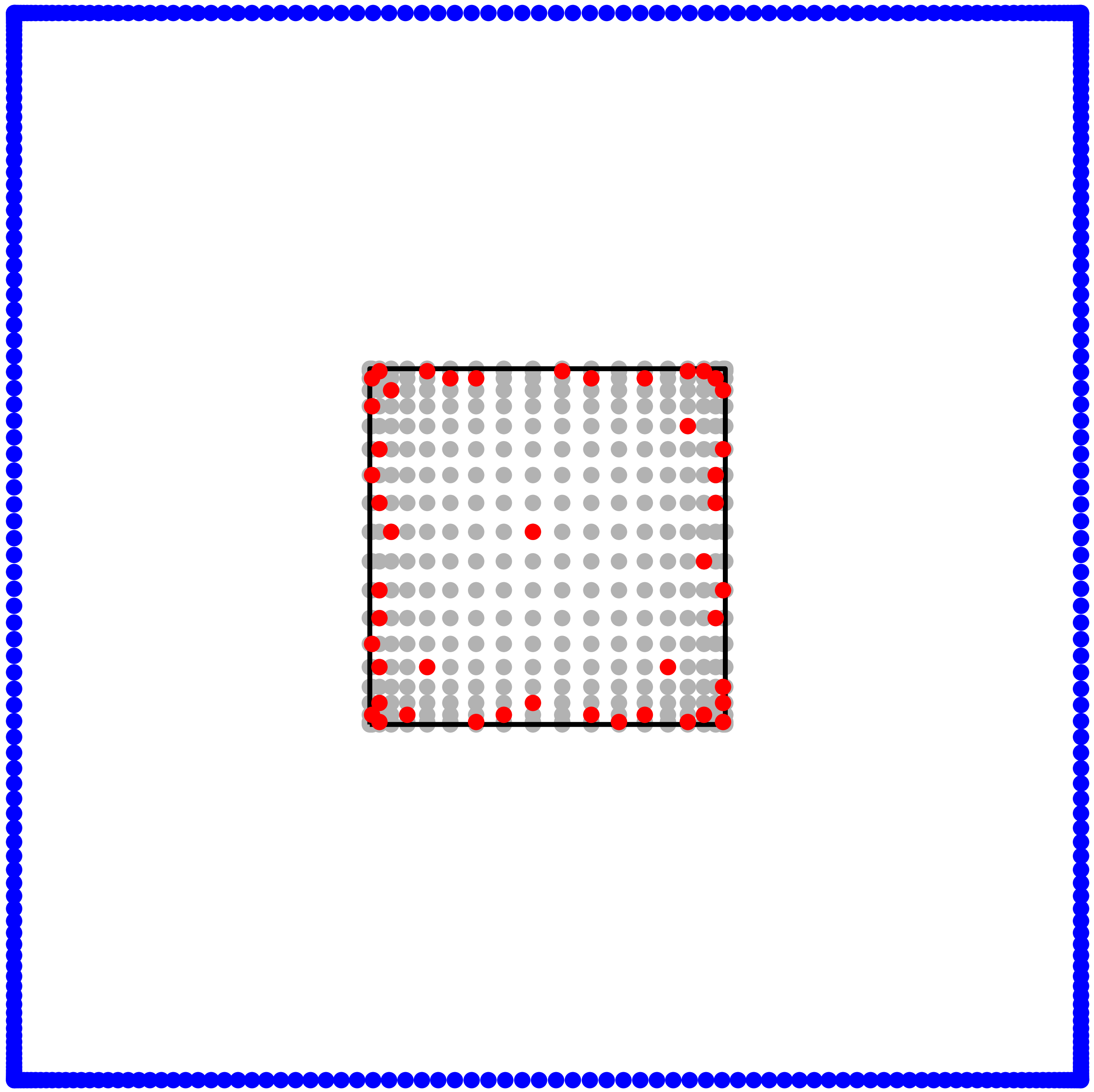
(The weirdness at the end reflects the discretization error.)

Example: Two squares — realistic FMM geometry.



Sources in a box of side length 1, targets on a box of side length 3.

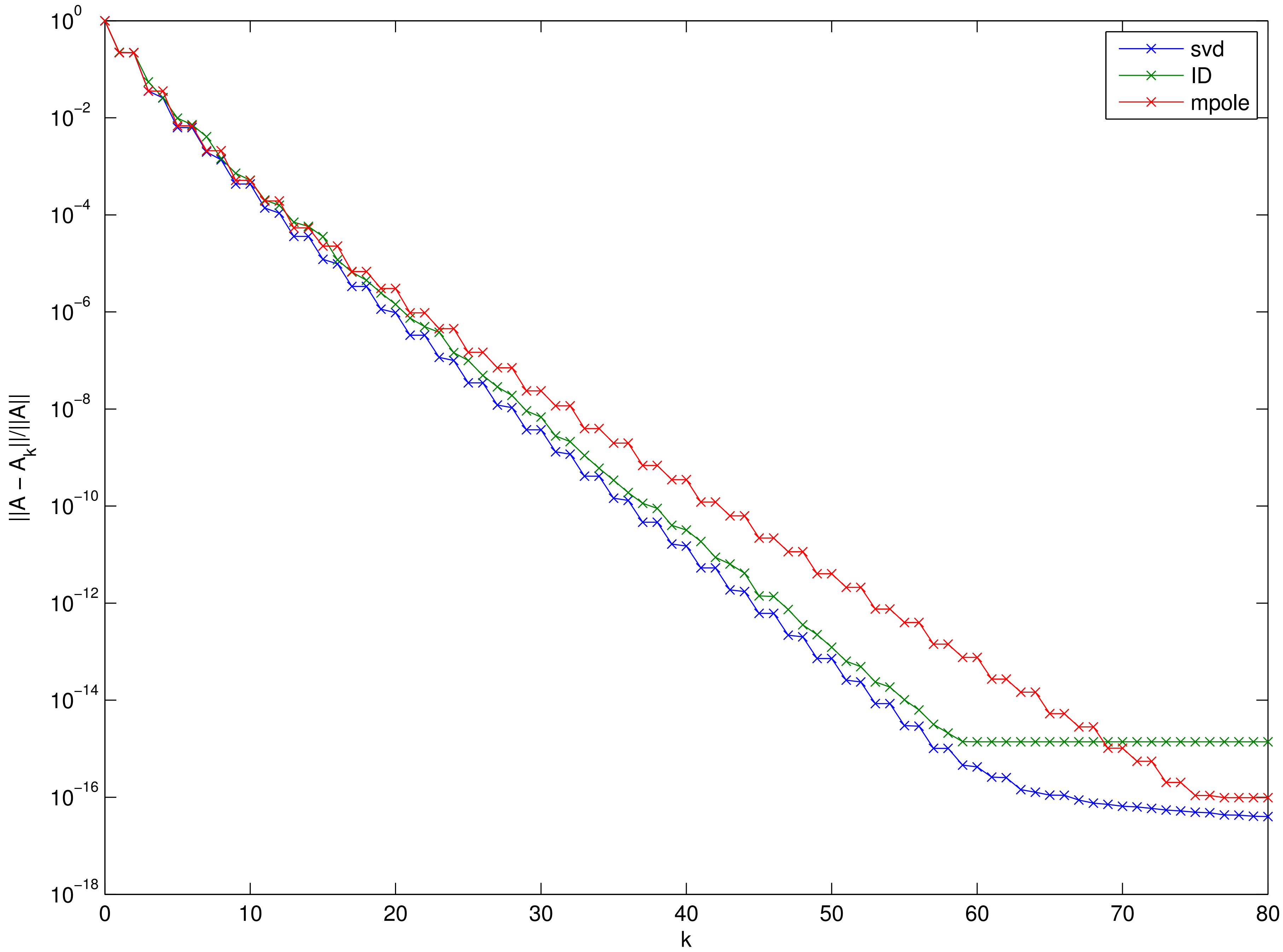
Example: Two squares — realistic FMM geometry.



Sources in a box of side length 1, targets on a box of side length 3.

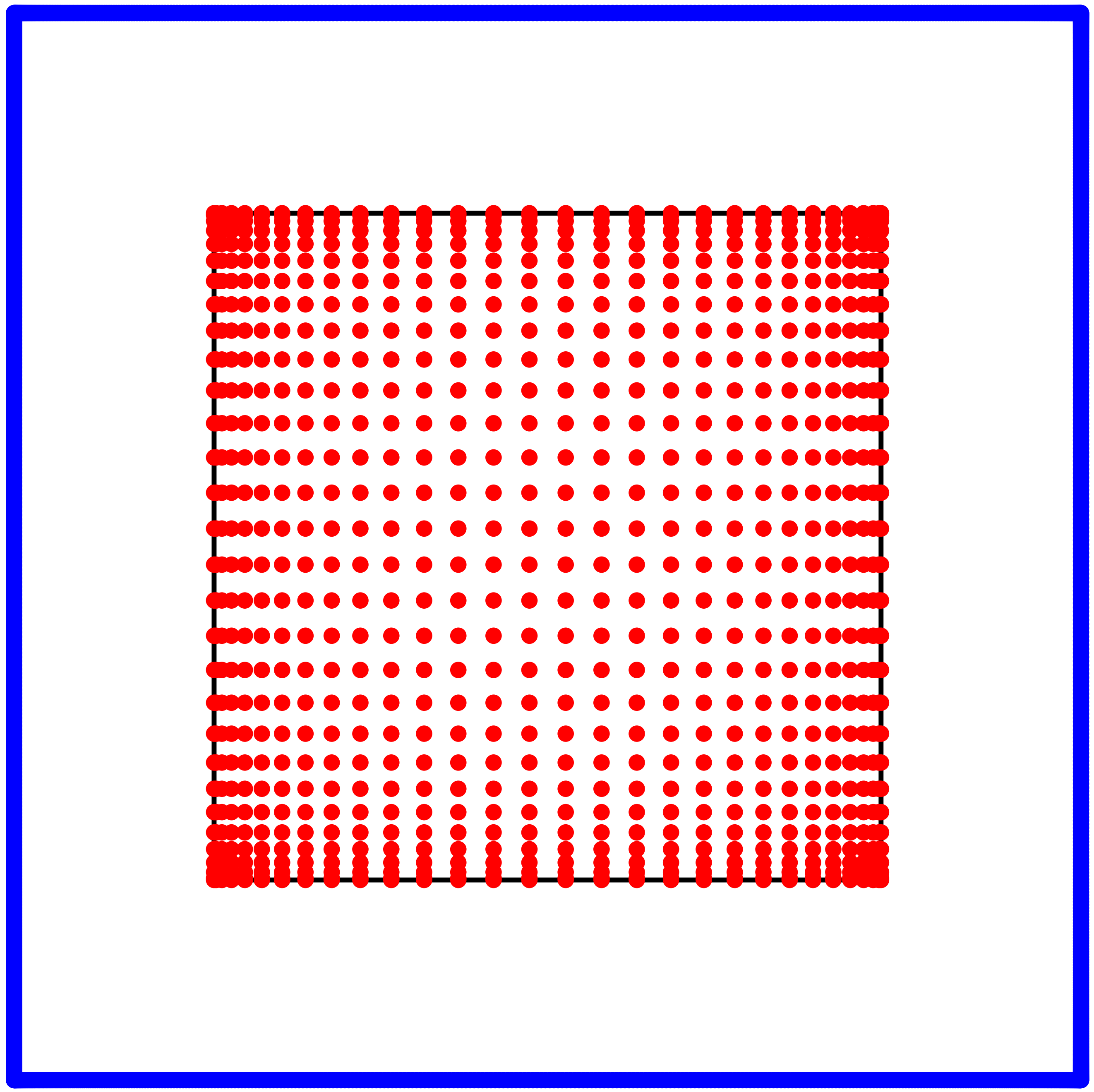
Skeleton to precision $\varepsilon = 10^{-12}$*, which requires* $k = 47$*.*

Example: Two squares — realistic FMM geometry.



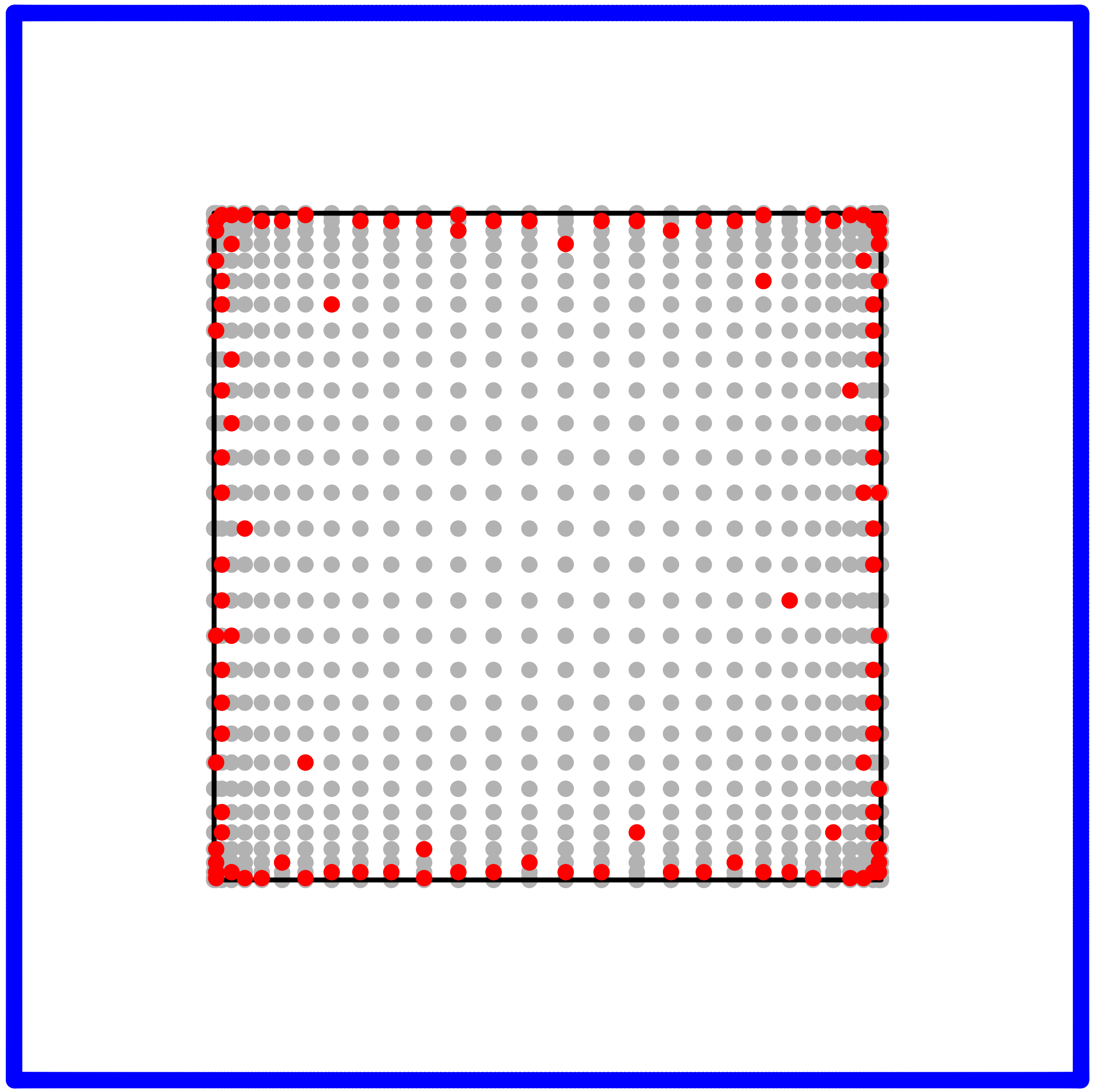
Errors.

Example: Two squares — now tighter.



Sources in a box of side length 1, targets on a box of side length 1.6.

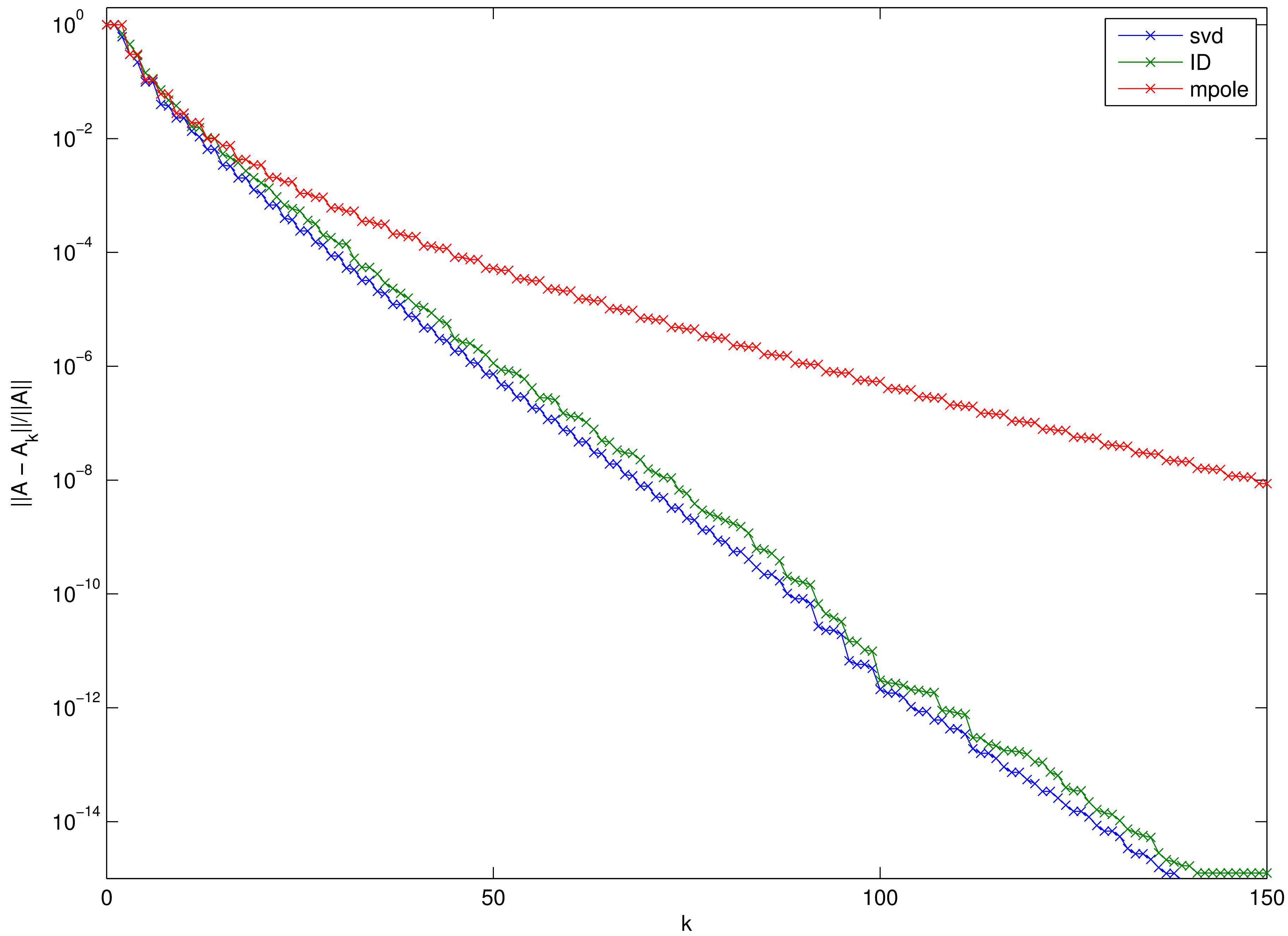
Example: Two squares — now tighter.



Sources in a box of side length 1, targets on a box of side length 1.6.

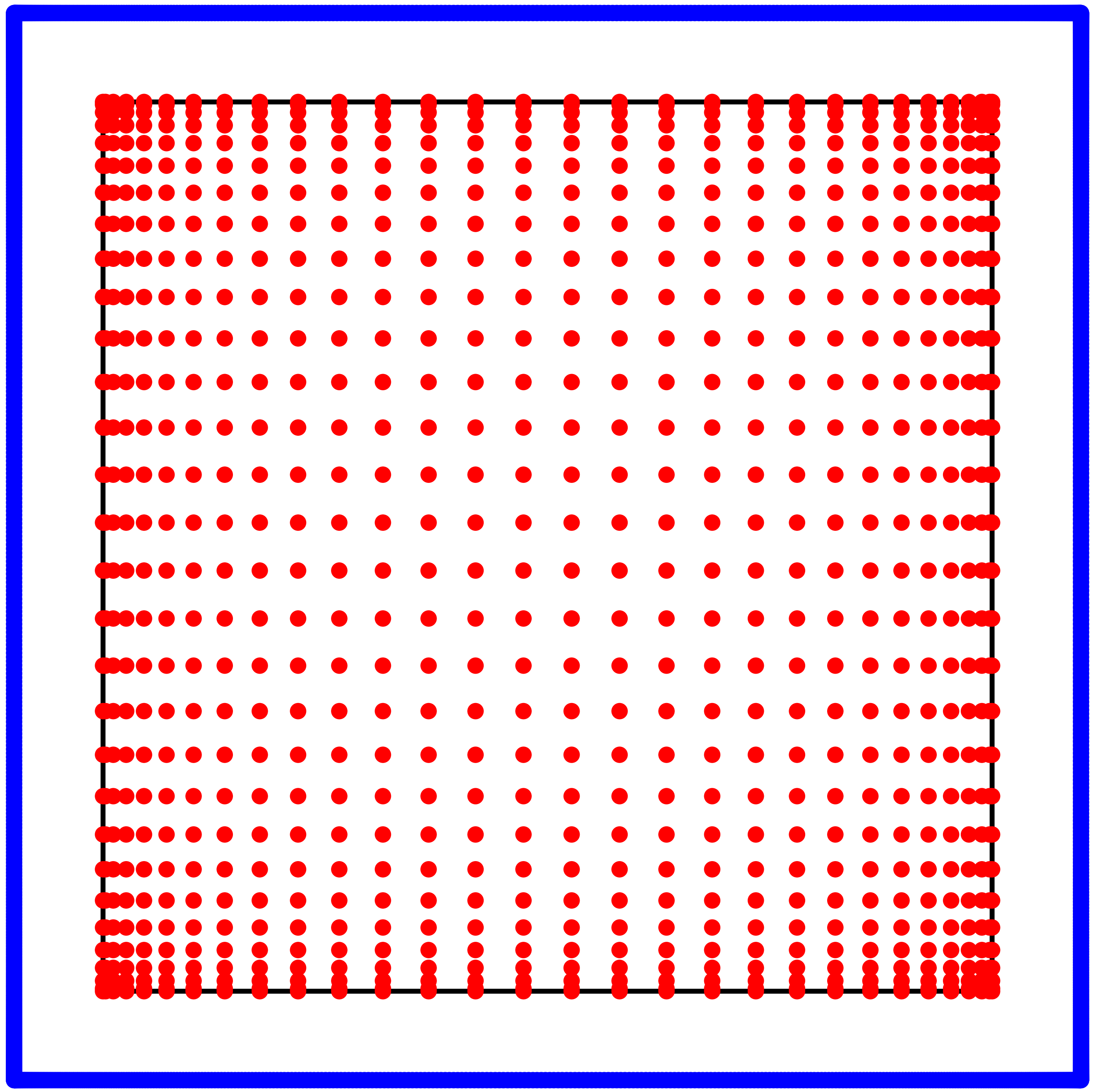
Skeleton to precision $\varepsilon = 10^{-12}$*, which requires* $k = 108$.

Example: Two squares — now tighter.



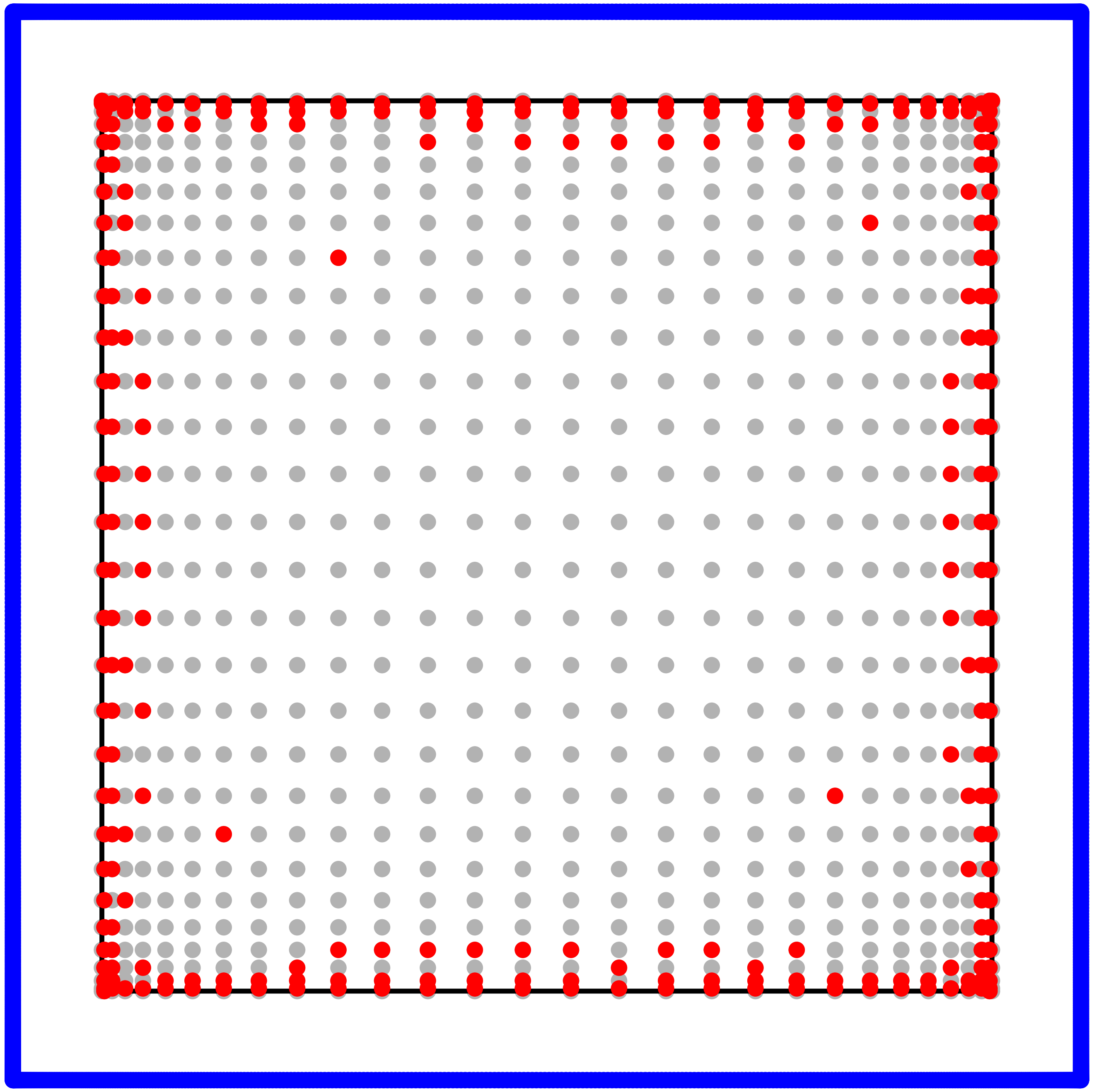
Sources in a box of side length 1, targets on a box of side length 1.6.

Example: Two squares — now even tighter.



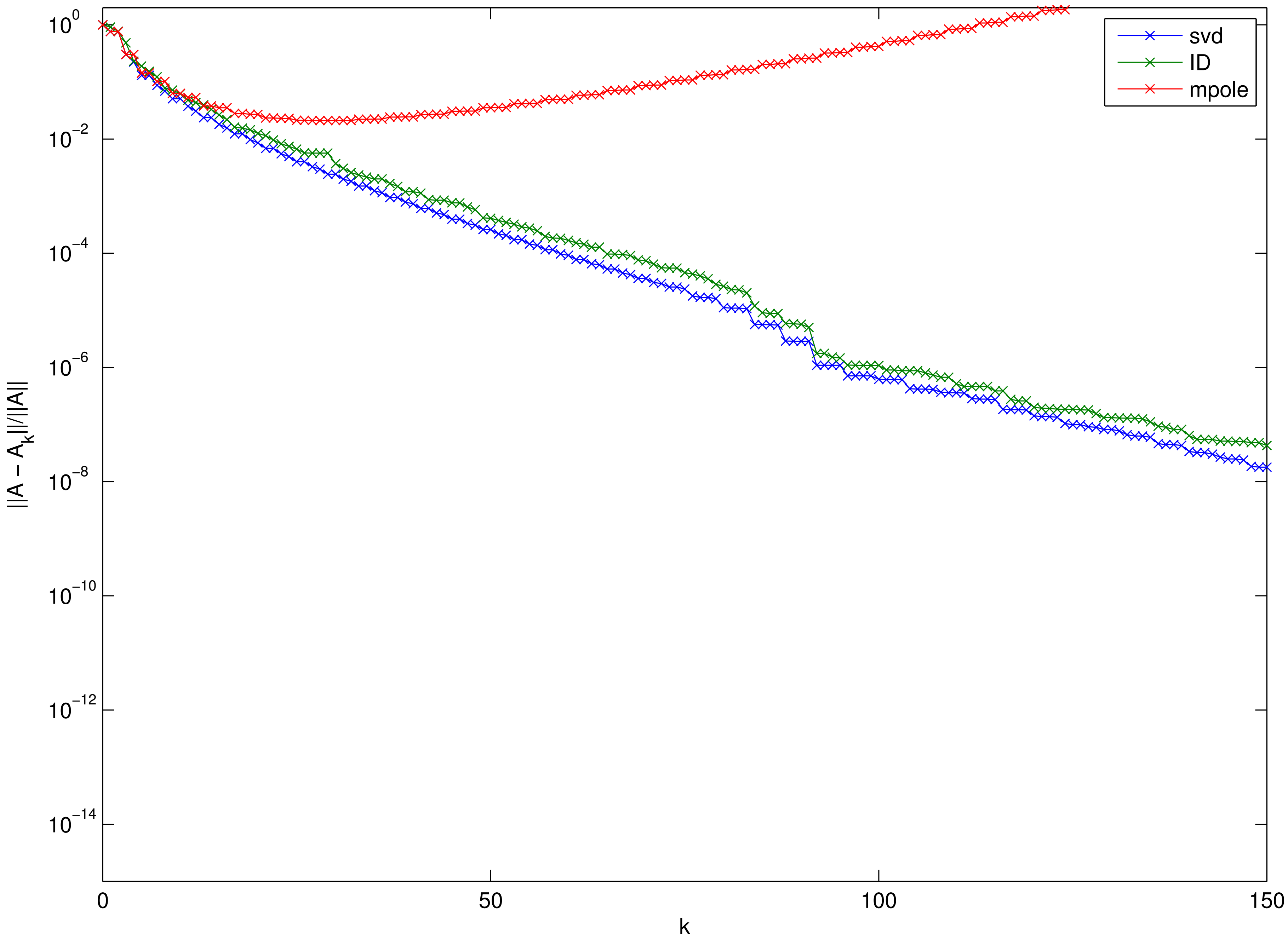
Sources in a box of side length 1, targets on a box of side length 1.2.

Example: Two squares — now even tighter.



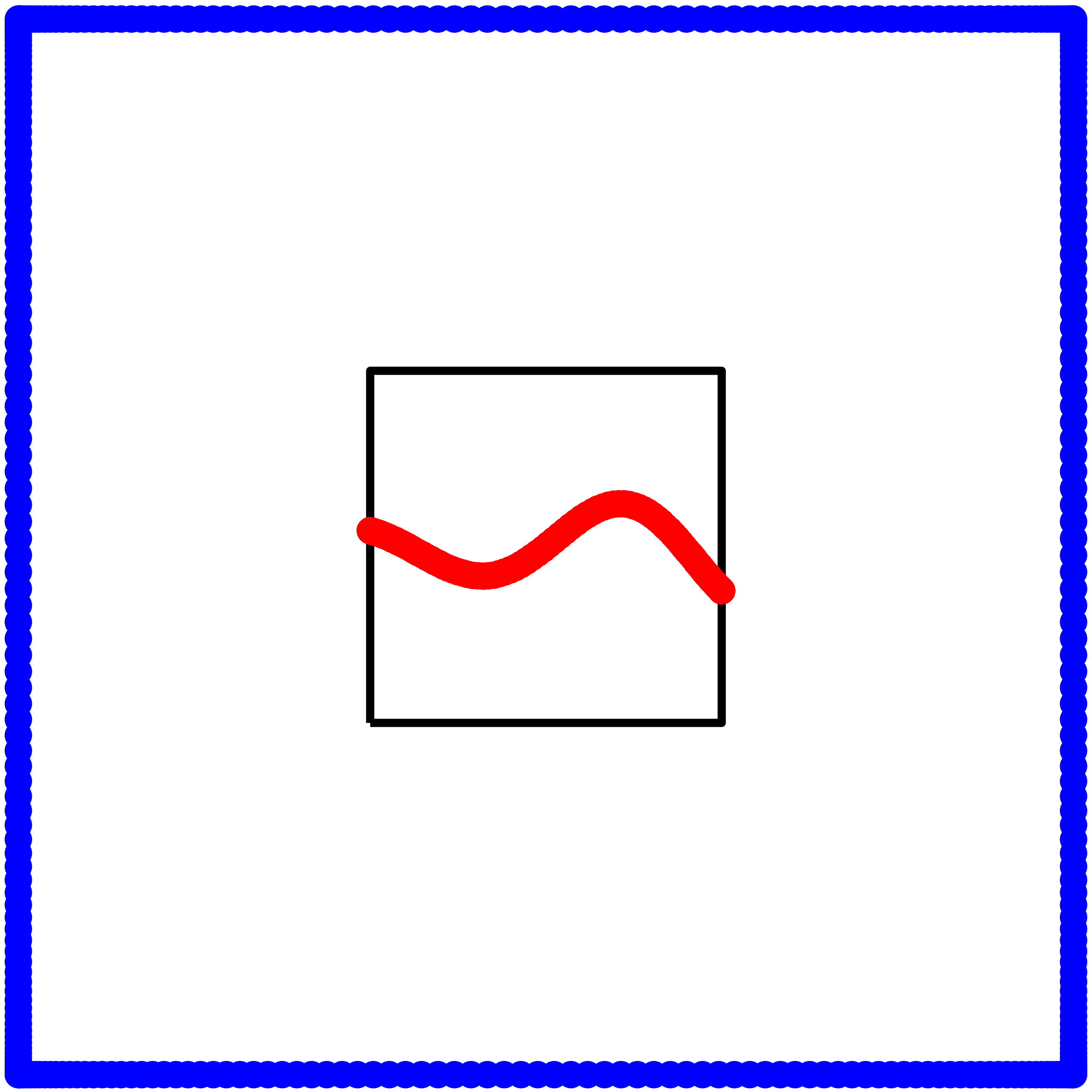
Sources in a box of side length 1, targets on a box of side length 1.2.
Skeleton to precision $\varepsilon = 10^{-12}$*, which requires* $k = 260$*.*

Example: Two squares — now even tighter.

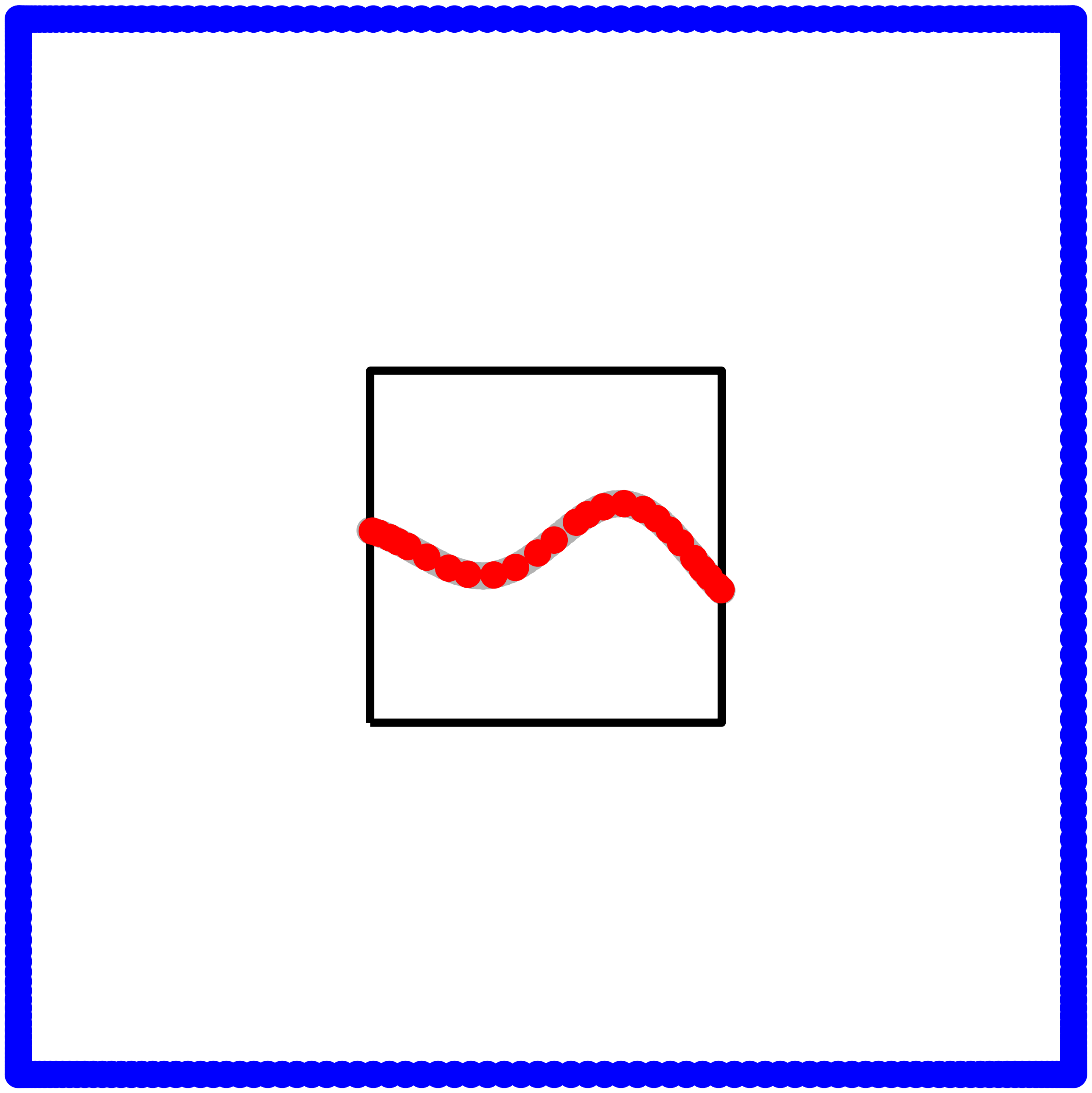


Sources in a box of side length 1, targets on a box of side length 1.2.

Example: A piece of a contour.

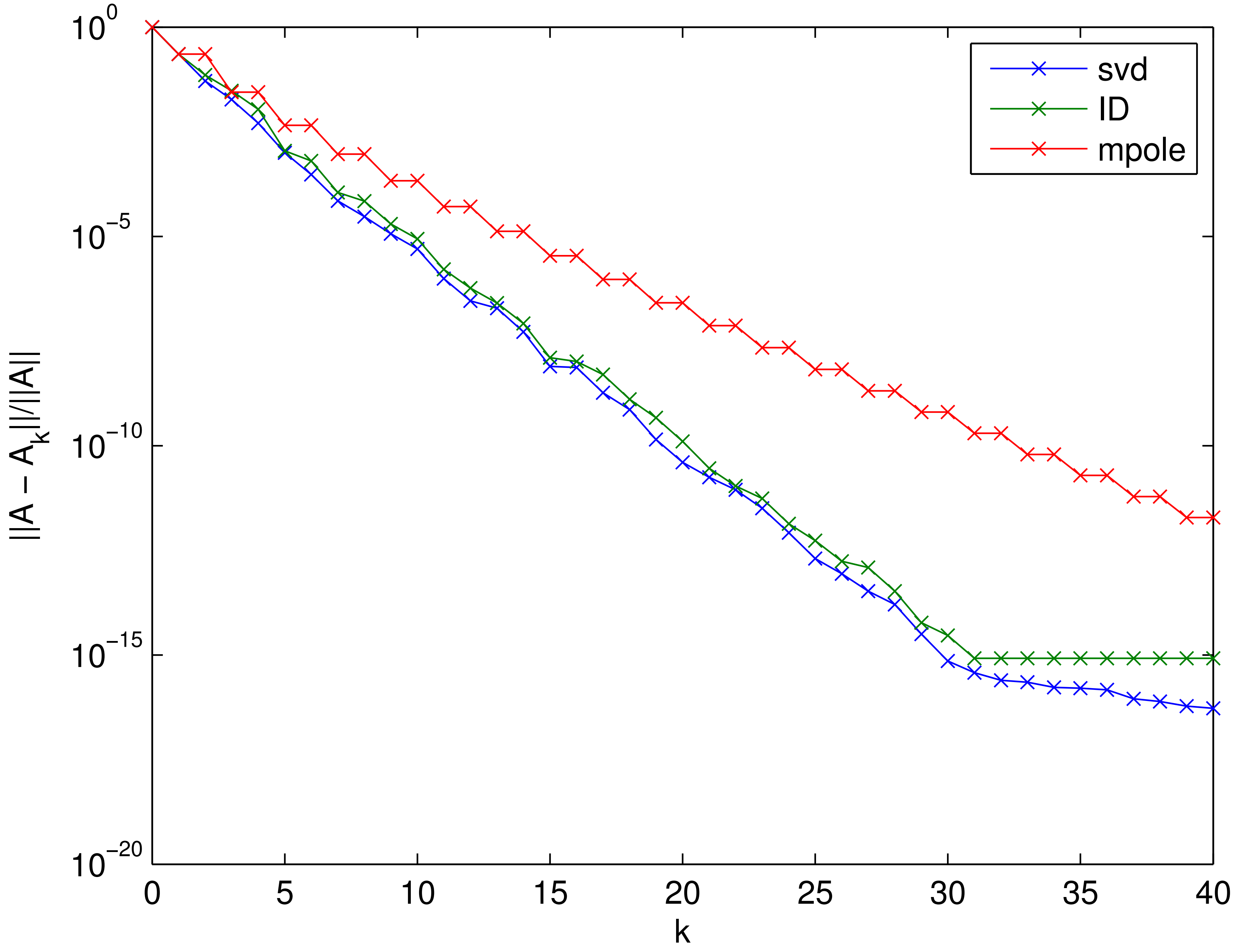


Example: A piece of a contour.

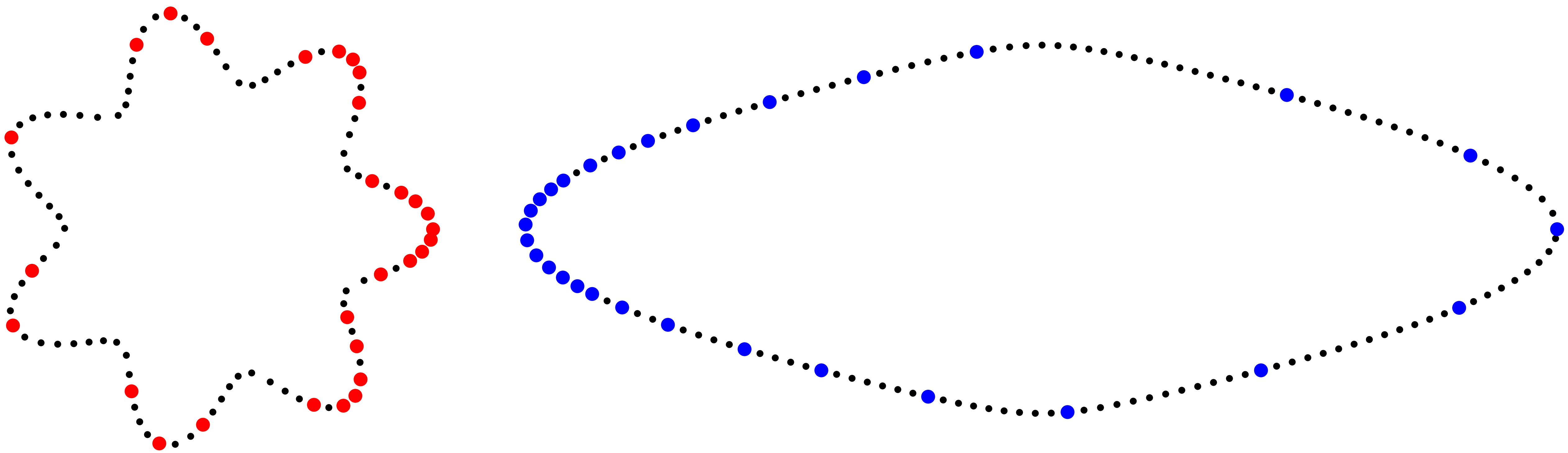


Skeleton to precision $\varepsilon = 10^{-12}$, which requires $k = 25$.

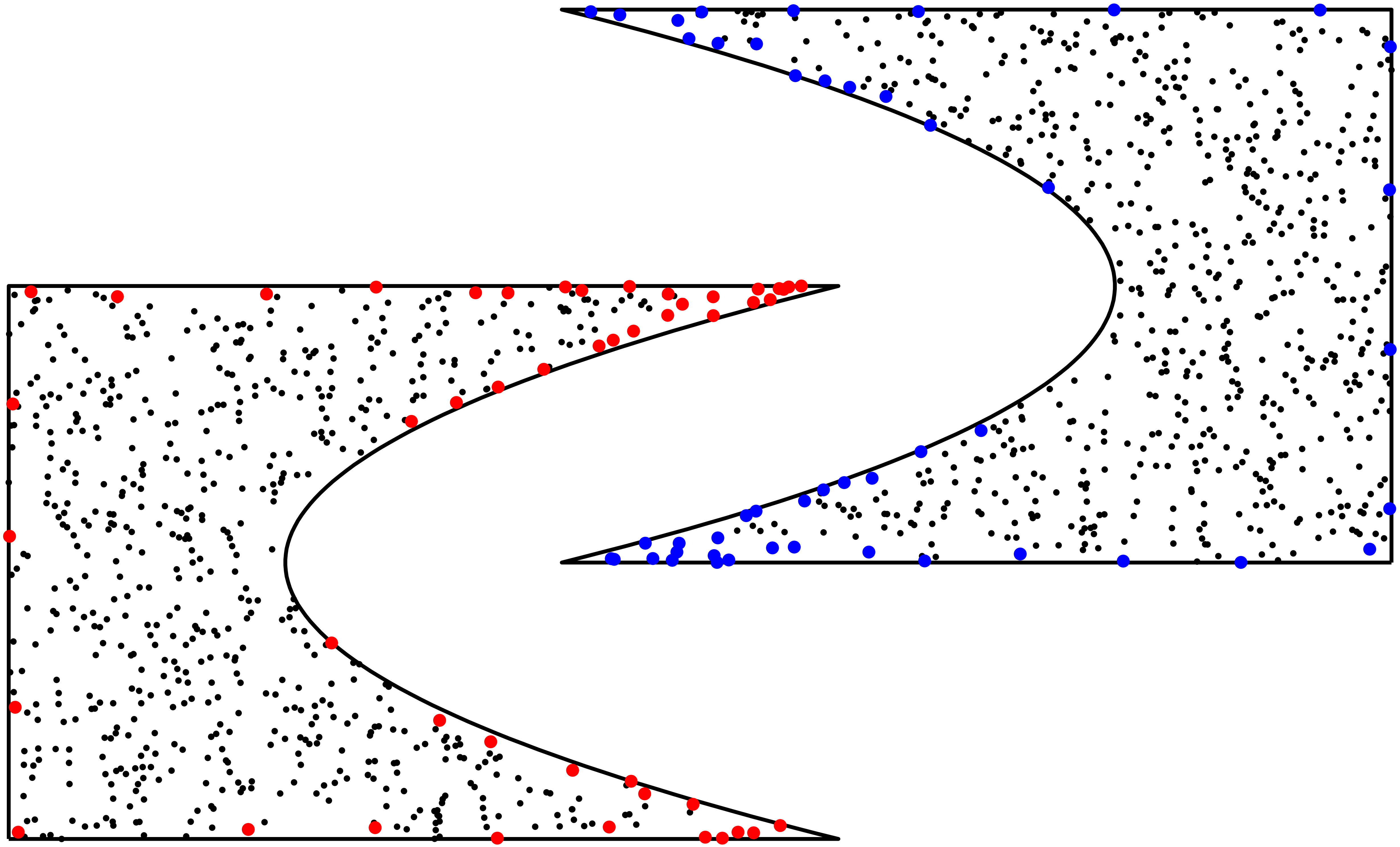
Example: A piece of a contour.



Skeletonization can be performed for Ω_{S} and Ω_{T} of various shapes.

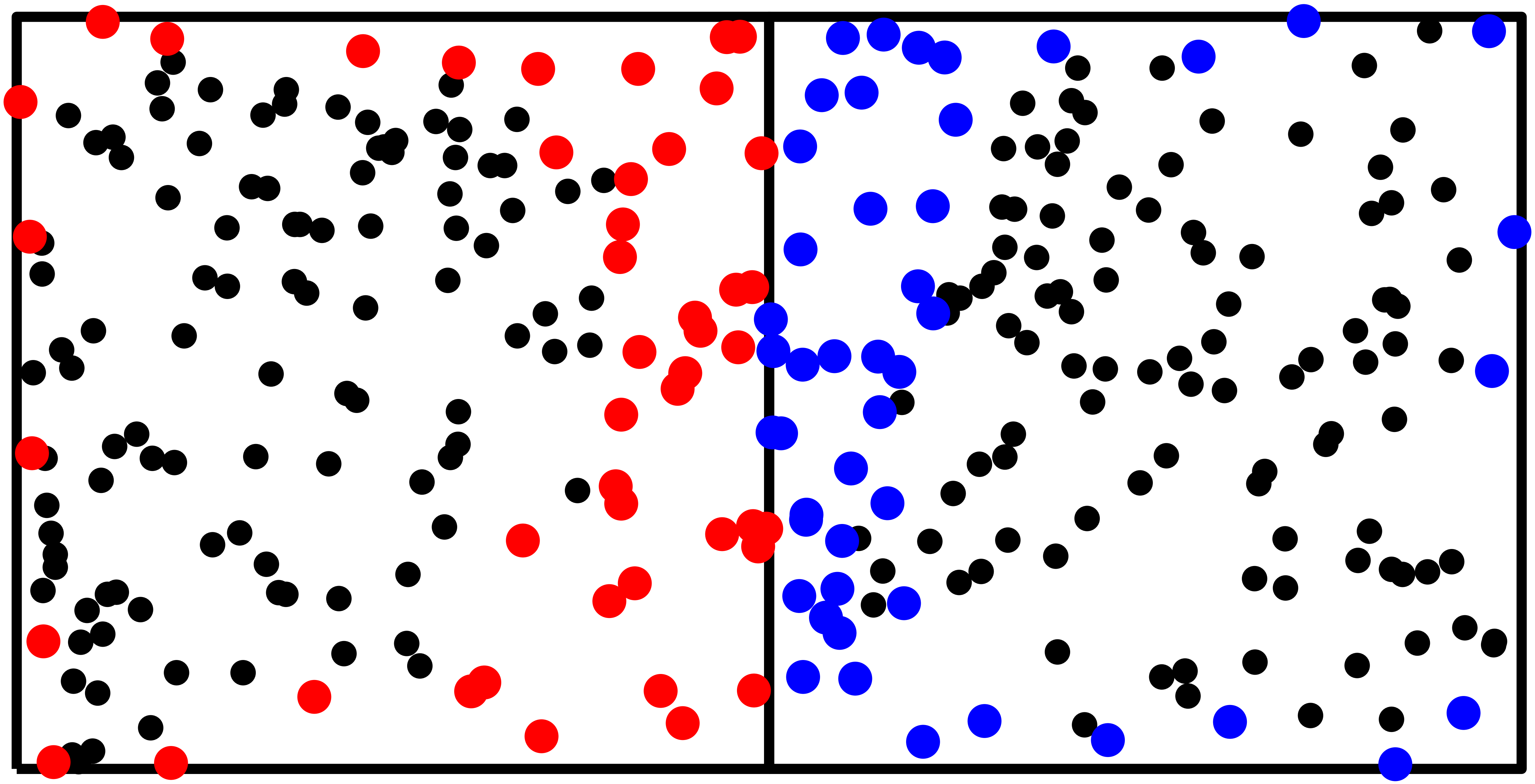


Rank = 29 at $\varepsilon = 10^{-10}$.

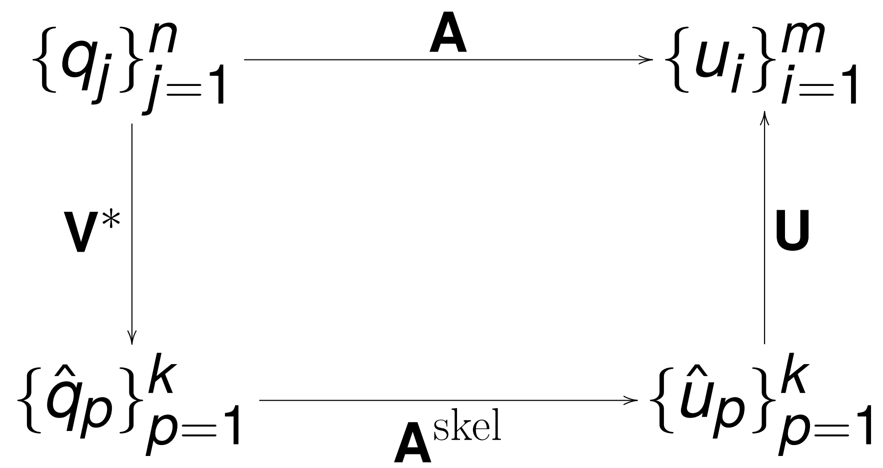


Rank = 48 at $\varepsilon = 10^{-10}$.

Adjacent boxes can be skeletonized.



Rank = 46 at $\varepsilon = 10^{-10}$.



Benefits:

- The rank is typically very close to optimal.
- The projection and interpolation are well-conditioned.
- An inexpensive local computation (e.g. Gram-Schmidt) determines:
  - The k skeleton points.
  - Matrices **U** and **V**.
- The map $\mathbf{A}^{\rm skel}$ has the same kernel as **A**.

  (We loosely say that “the physics of the problem is preserved”.)
- The skeleton points can be determined either as generic points valid for any source distribution, or as a subset of a given set of points. In the latter case **U** and **V** contain $k \times k$ identity matrices.
- Interaction between *adjacent* boxes can be compressed (no buffering is required).

Before closing this topic, let us briefly consider the *Helmholtz problem.*

Recall that the Helmholtz equation is associated with the classical wave equation

$$-v^2\Delta\phi = -\frac{\partial^2\phi}{\partial t^2}, \tag{2}$$

where v is the wave-speed. Assume $\phi(\boldsymbol{x}, t) = u(\boldsymbol{x})\, e^{i\omega t}$. Then (2) turns into

$$-v^2\Delta u = \omega^2 u, \tag{3}$$

We define the "wave number" as $\kappa = \omega/v$, and can then write (3) as

$$-\Delta u - \kappa^2 u = 0. \tag{4}$$

A typical "free-space" problem for the Helmholtz equation could read

$$\begin{cases} -\Delta u(\boldsymbol{x}) - \kappa^2 u(\boldsymbol{x}) = q(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^2 \\ \dfrac{\partial u(\boldsymbol{x})}{\partial |\boldsymbol{x}|} - i\kappa u(\boldsymbol{x}) = O\left(\dfrac{1}{|\boldsymbol{x}|}\right) & |\boldsymbol{x}| \to \infty, \end{cases} \tag{5}$$

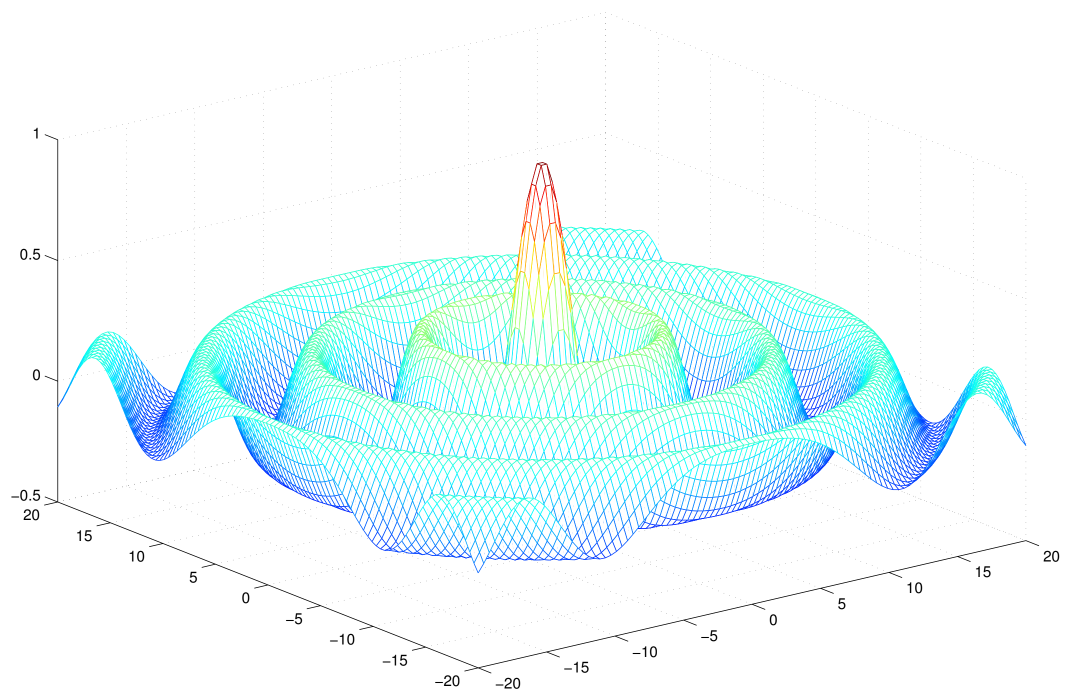
where the condition "at infinity" is called a "radiation condition."

We typically consider u to be a *complex valued* potential.

The fundamental solution is $H_0^{(1)}(\kappa|\boldsymbol{x}|)$, so the solution to (5) is

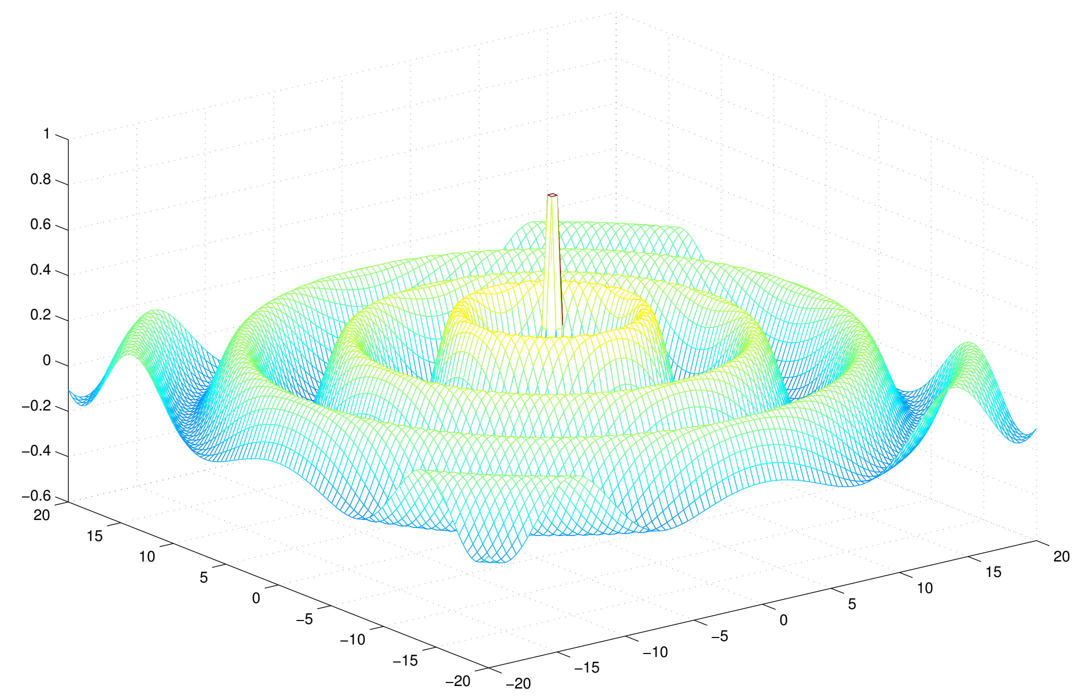
$$u(\boldsymbol{x}) = \int_{\mathbb{R}^2} H_0^{(1)}(\kappa|\boldsymbol{x}-\boldsymbol{y}|)\, q(\boldsymbol{y})\, d\boldsymbol{y}.$$

Plots of the fundamental solution $H_0^{(1)}(|\boldsymbol{x}|) = J_0(|\boldsymbol{x}|) + i\,Y_0(|\boldsymbol{x}|)$.



Real part

J_0

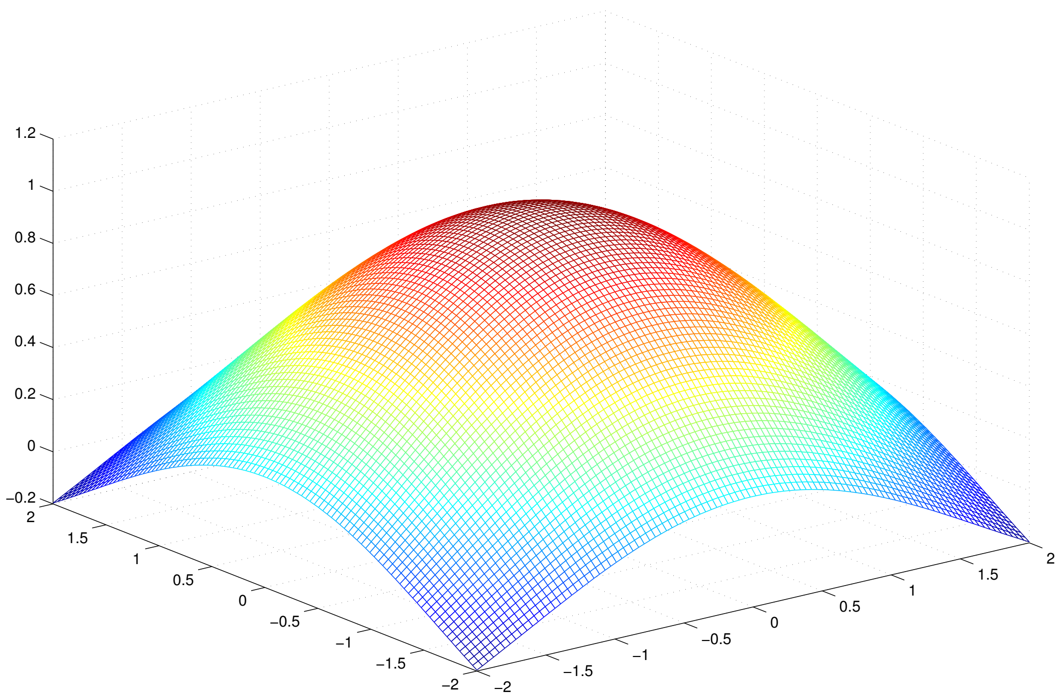


Negative imaginary part

$-Y_0$

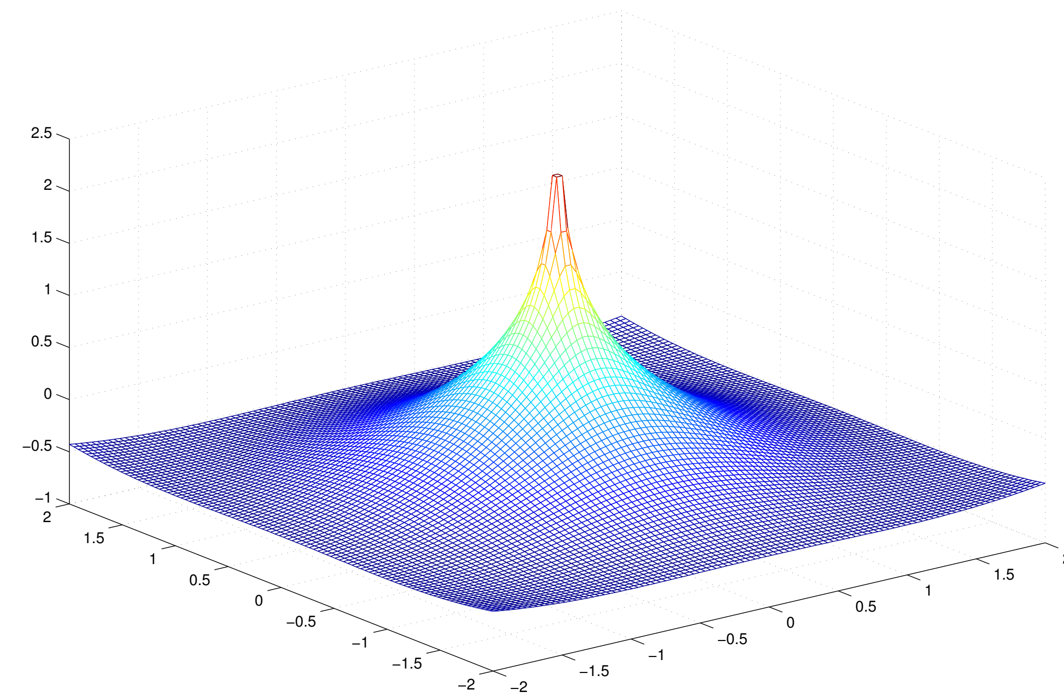
Plots of the fundamental solution $H_0^{(1)}(|\boldsymbol{x}|) = J_0(|\boldsymbol{x}|) + i\,Y_0(|\boldsymbol{x}|)$.

Now zoom in to the origin:



Real part

J_0



Negative imaginary part

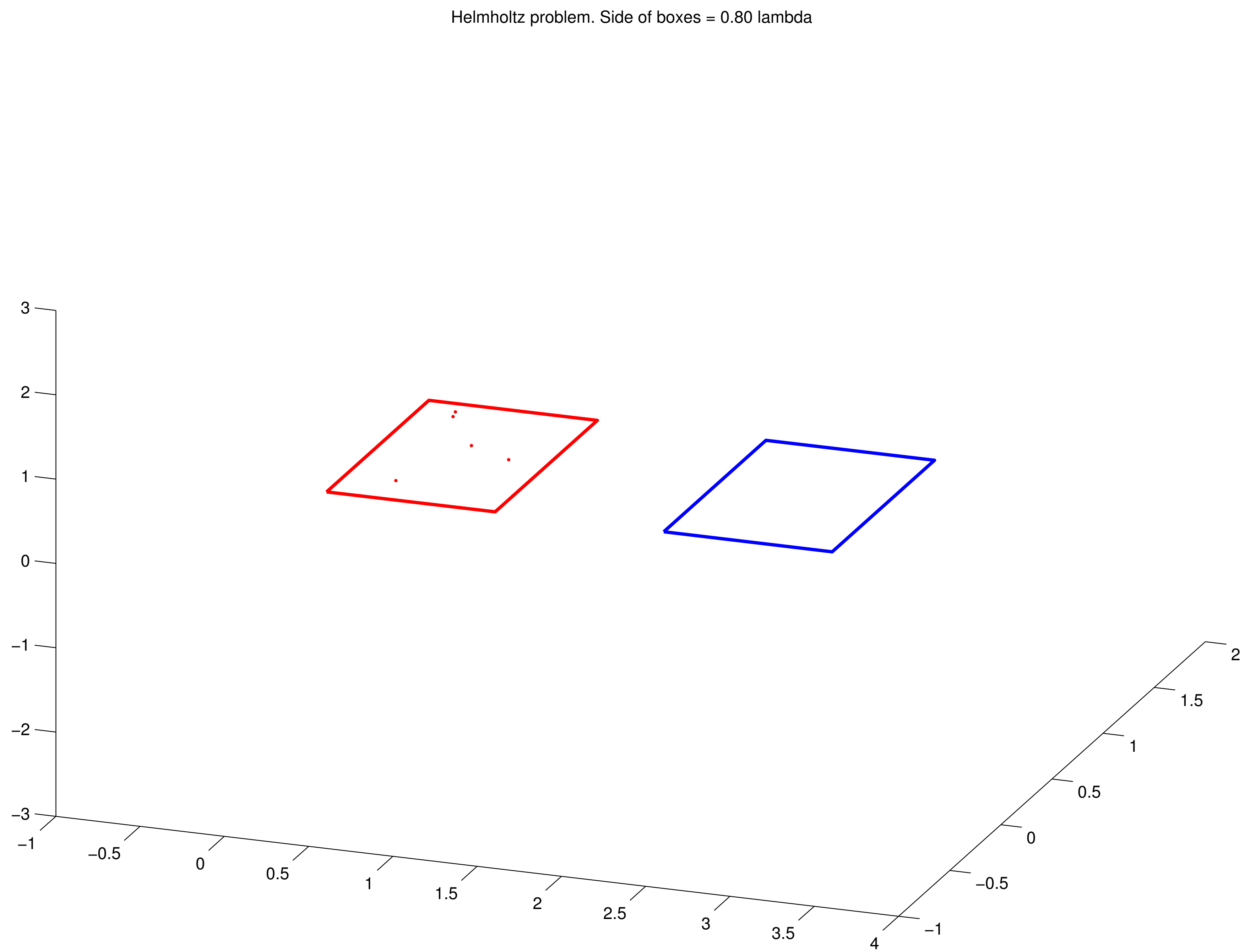
$-Y_0$

Logarithmic singularity at the origin!

Example of solution of the Helmholtz equation $-\Delta u - \kappa^2 u = g$

Suppose we are given point charges $\{q_j\}_{j=1}^{5}$ in a "source domain" $\Omega_{\rm s}$.

We are interested in the potential in a "target domain" $\Omega_{\rm t}$.



The source domain $\Omega_{\rm s}$ (red) and the target domain $\Omega_{\rm t}$ (blue).

Example of solution of the Helmholtz equation $-\Delta u - \kappa^2 u = g$

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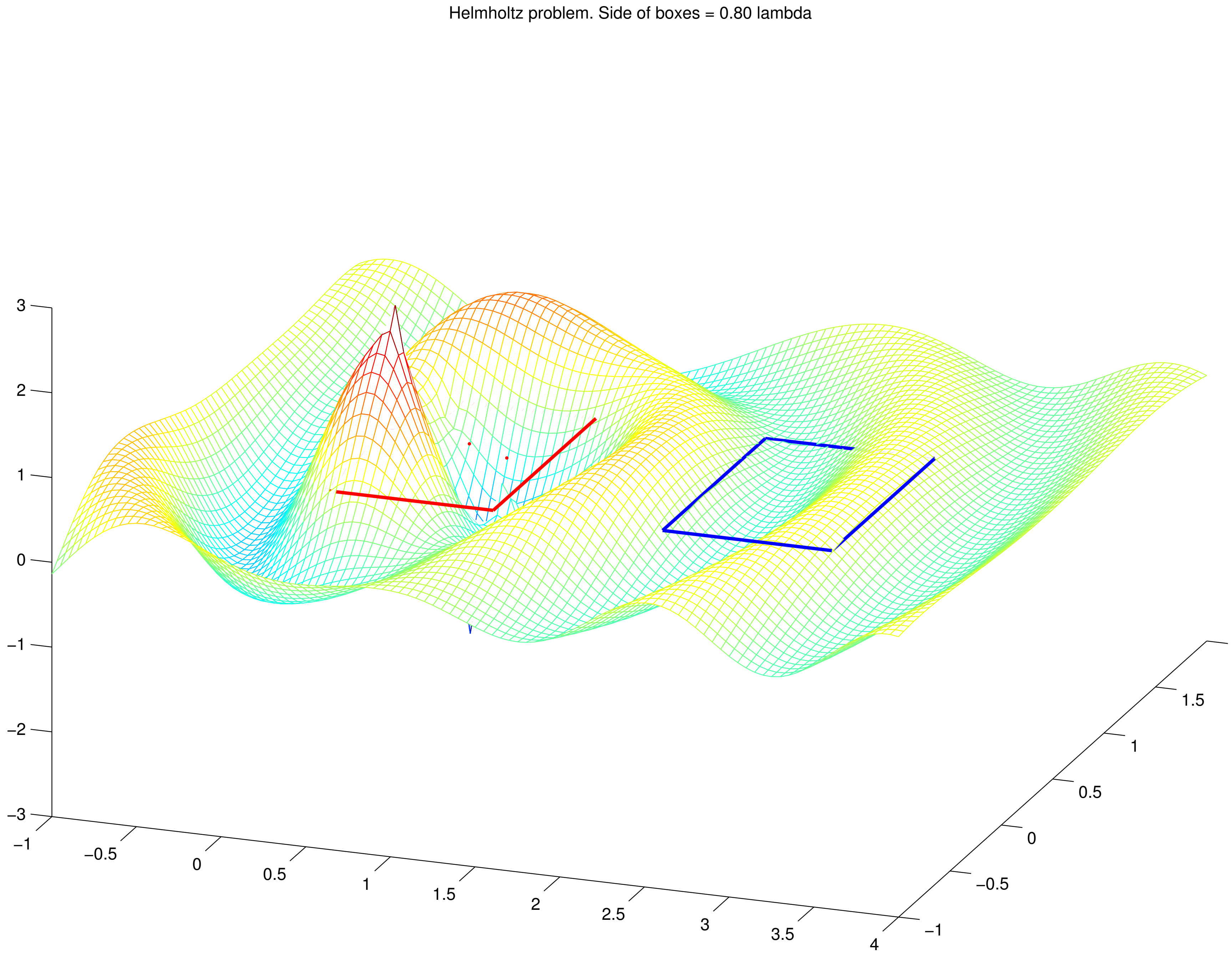
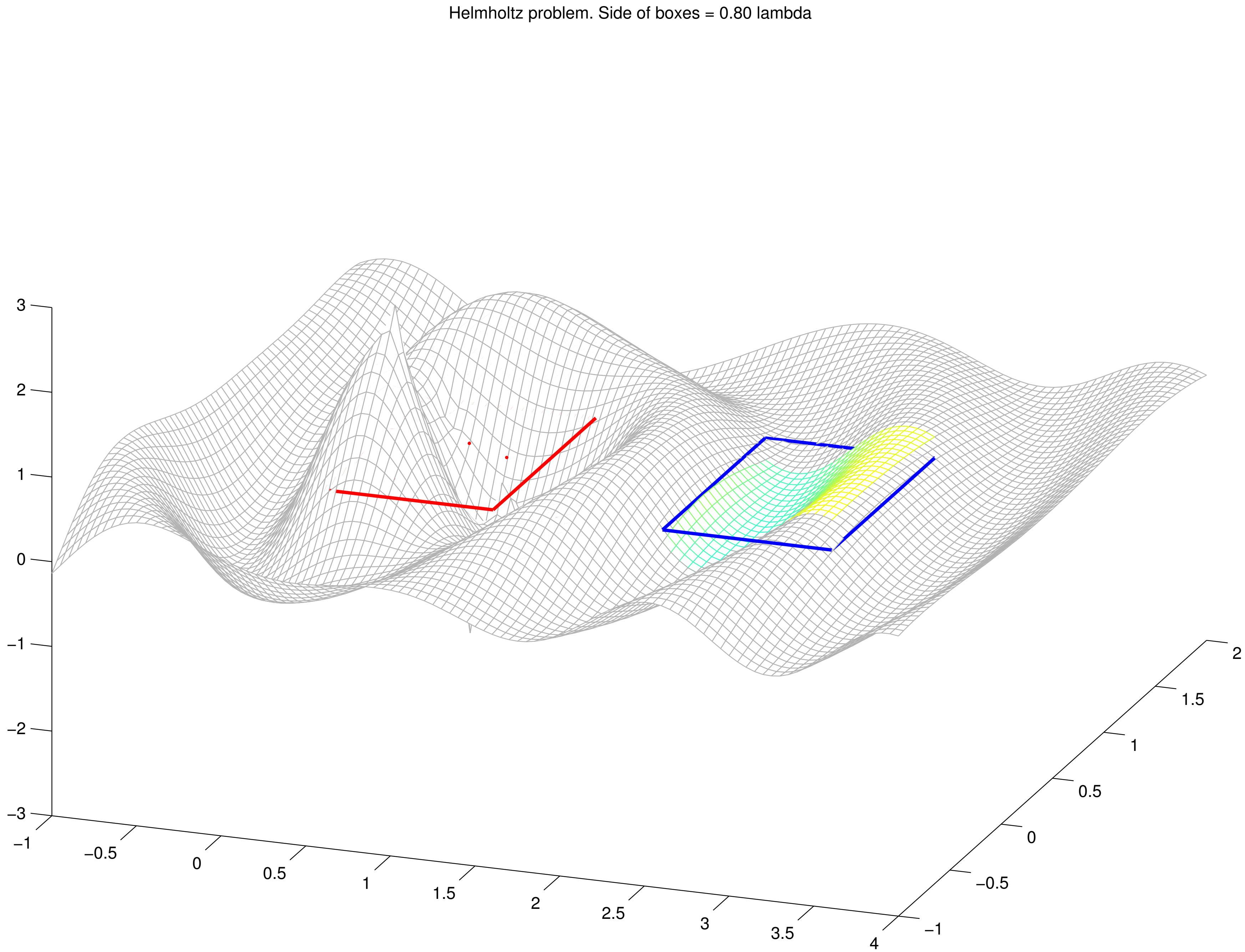


Real part of field generated by the sources (truncated — the peaks go to infinity).

Example of solution of the Helmholtz equation $-\Delta u - \kappa^2 u = g$

Suppose we are given point charges $\{q_j\}_{j=1}^{5}$ in a "source domain" $\Omega_{\rm s}$.

We are interested in the potential in a "target domain" $\Omega_{\rm t}$.



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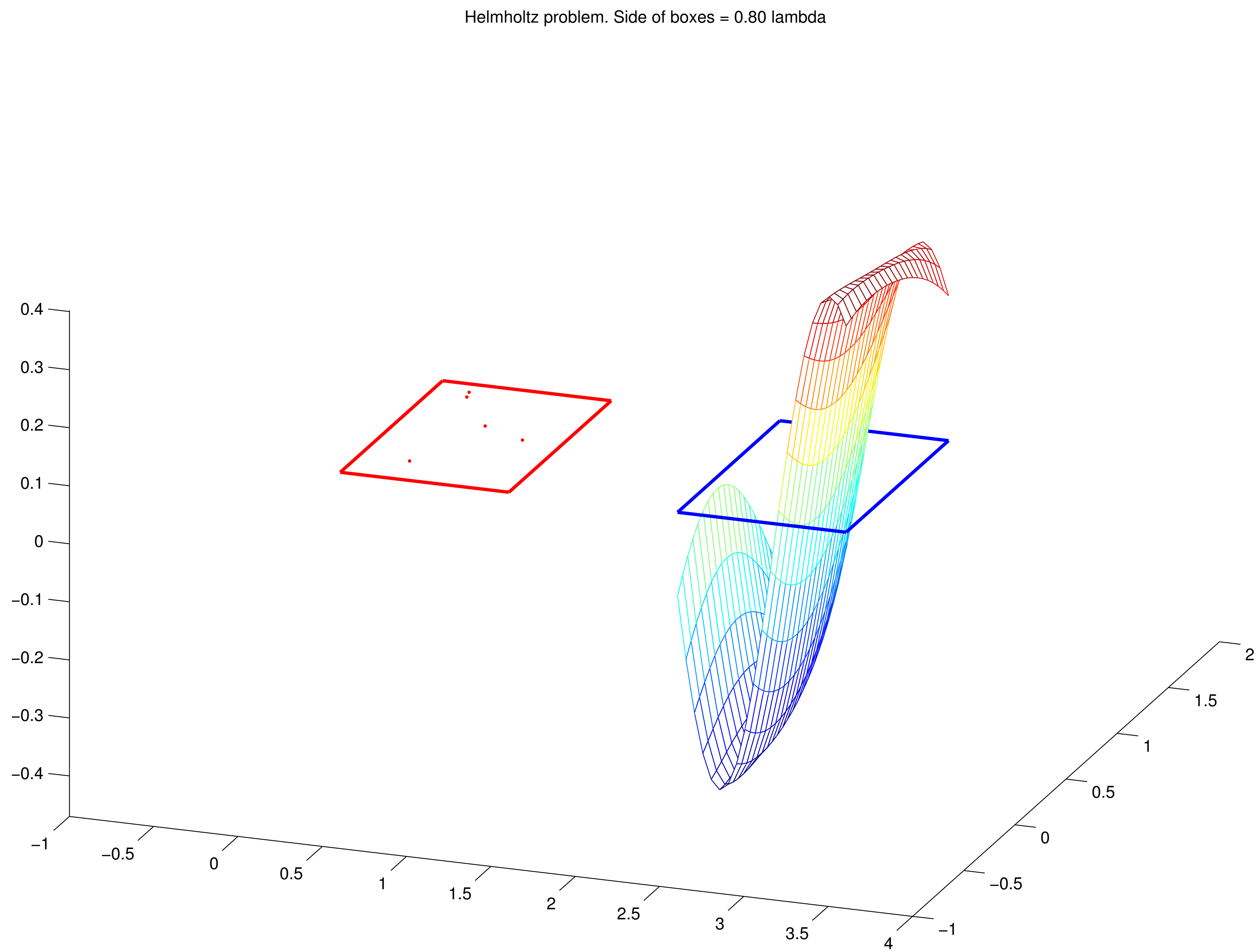
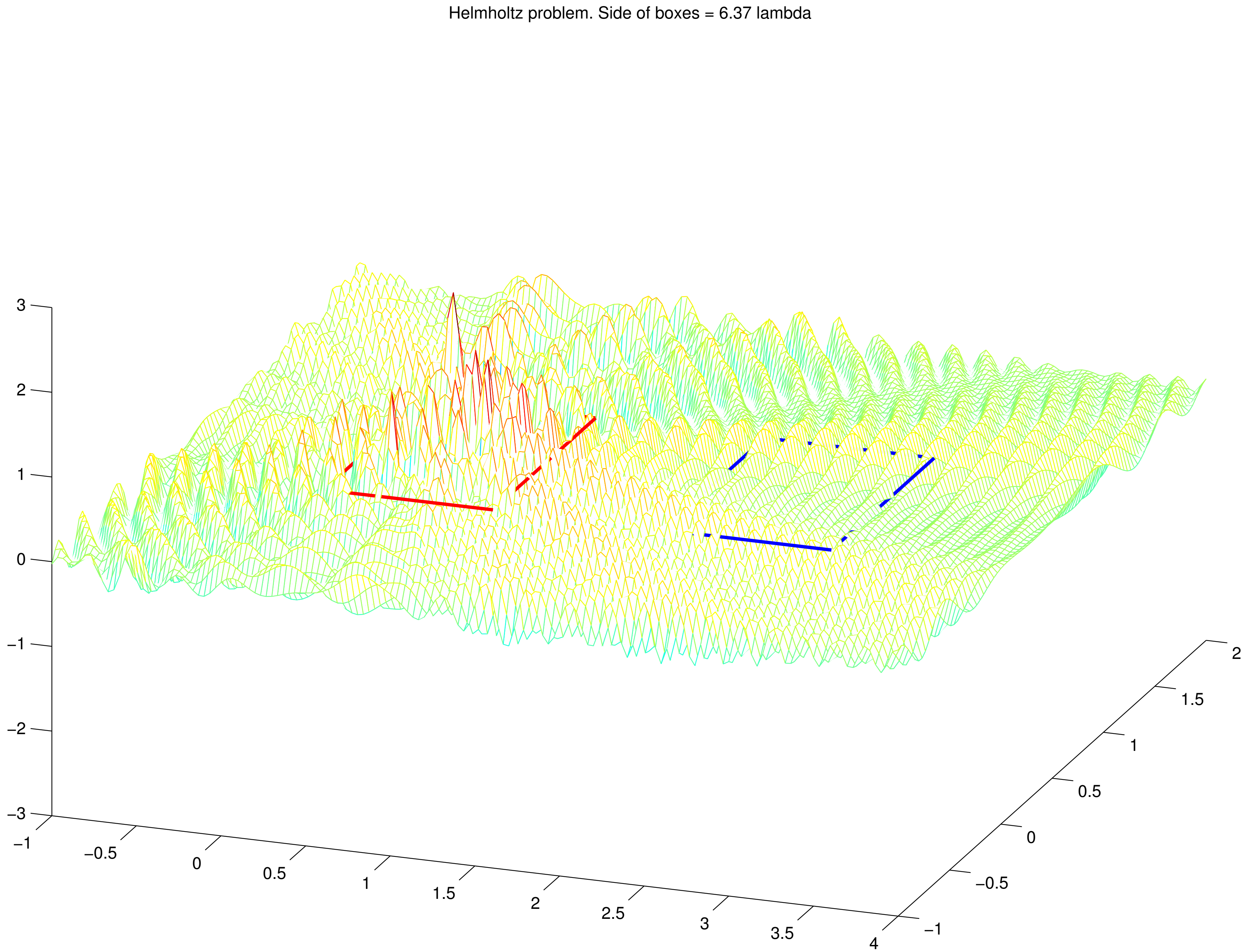


Real part of field generated by the sources (truncated — the peaks go to infinity).

Example of solution of the Helmholtz equation $-\Delta u - \kappa^2 u = g$

Suppose we are given point charges $\{q_j\}_{j=1}^{5}$ in a "source domain" $\Omega_{\rm s}$.

We are interested in the potential in a "target domain" $\Omega_{\rm t}$. *Now for larger κ!*



Real part of field generated by the sources (truncated — the peaks go to infinity).

Example of solution of the Helmholtz equation $-\Delta u - \kappa^2 u = g$

Suppose we are given point charges $\{q_j\}_{j=1}^5$ in a "source domain" $\Omega_{\rm s}$.

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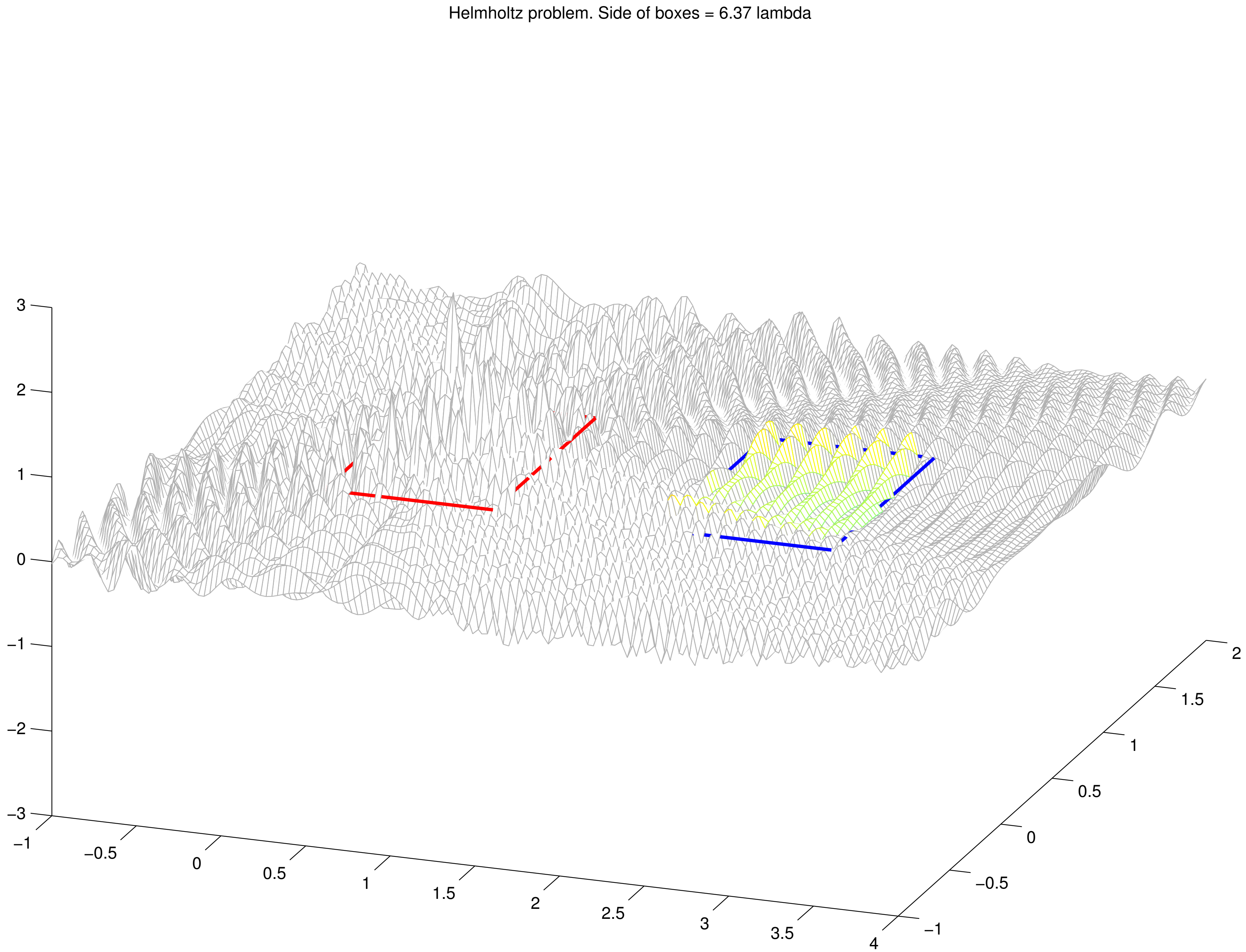
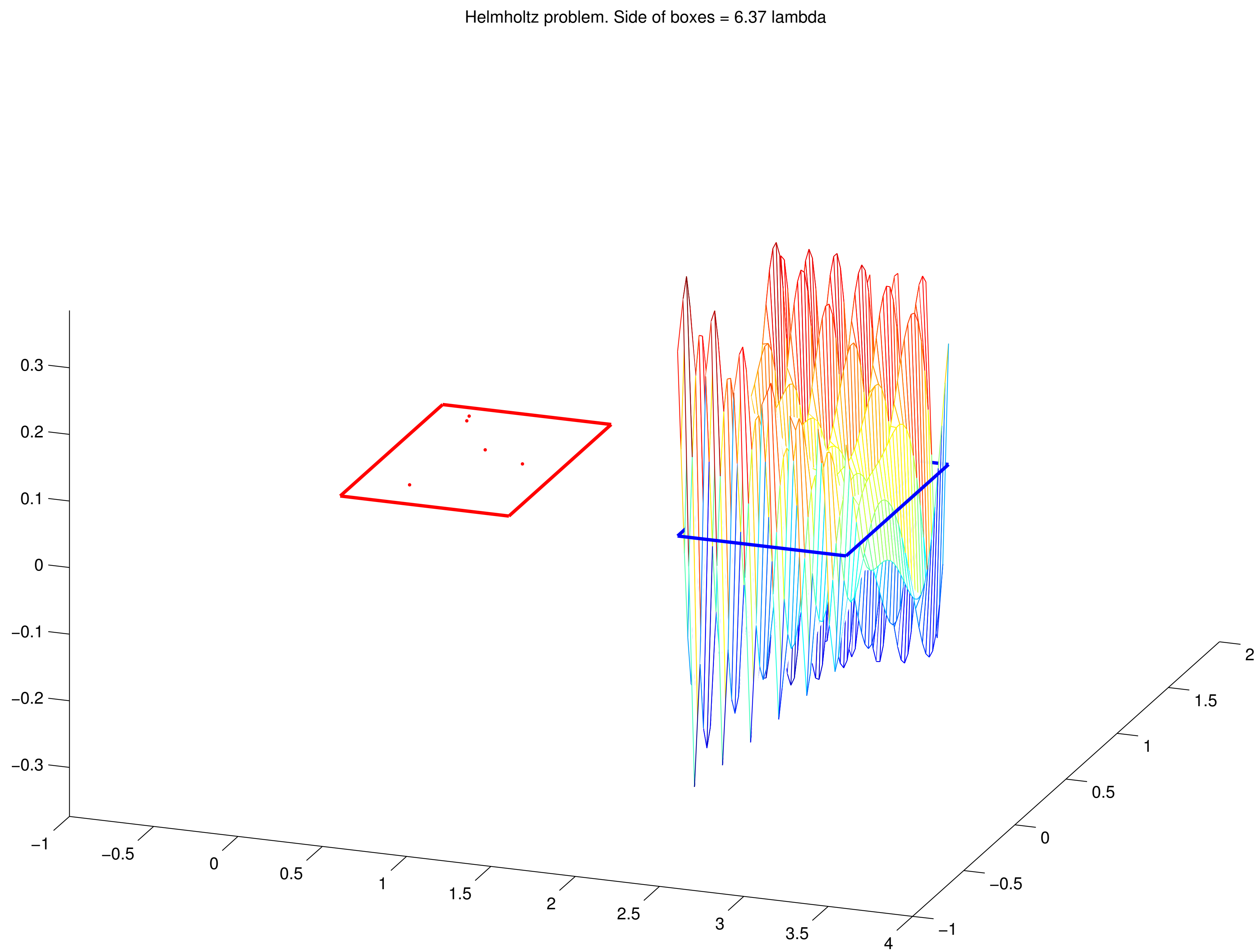


Real part of field generated by the sources (truncated — the peaks go to infinity).

Example of solution of the Helmholtz equation $-\Delta u - \kappa^2 u = g$

Suppose we are given point charges $\{q_j\}_{j=1}^{5}$ in a "source domain" $\Omega_{\rm s}$.

We are interested in the potential in a "target domain" $\Omega_{\rm t}$. *Now for larger κ!*



Real part of field generated by the sources (truncated — the peaks go to infinity).

Example of solution of the Helmholtz equation $-\Delta u - \kappa^2 u = g$

Suppose we are given point charges $\{q_j\}_{j=1}^5$ in a "source domain" $\Omega_{\rm s}$.

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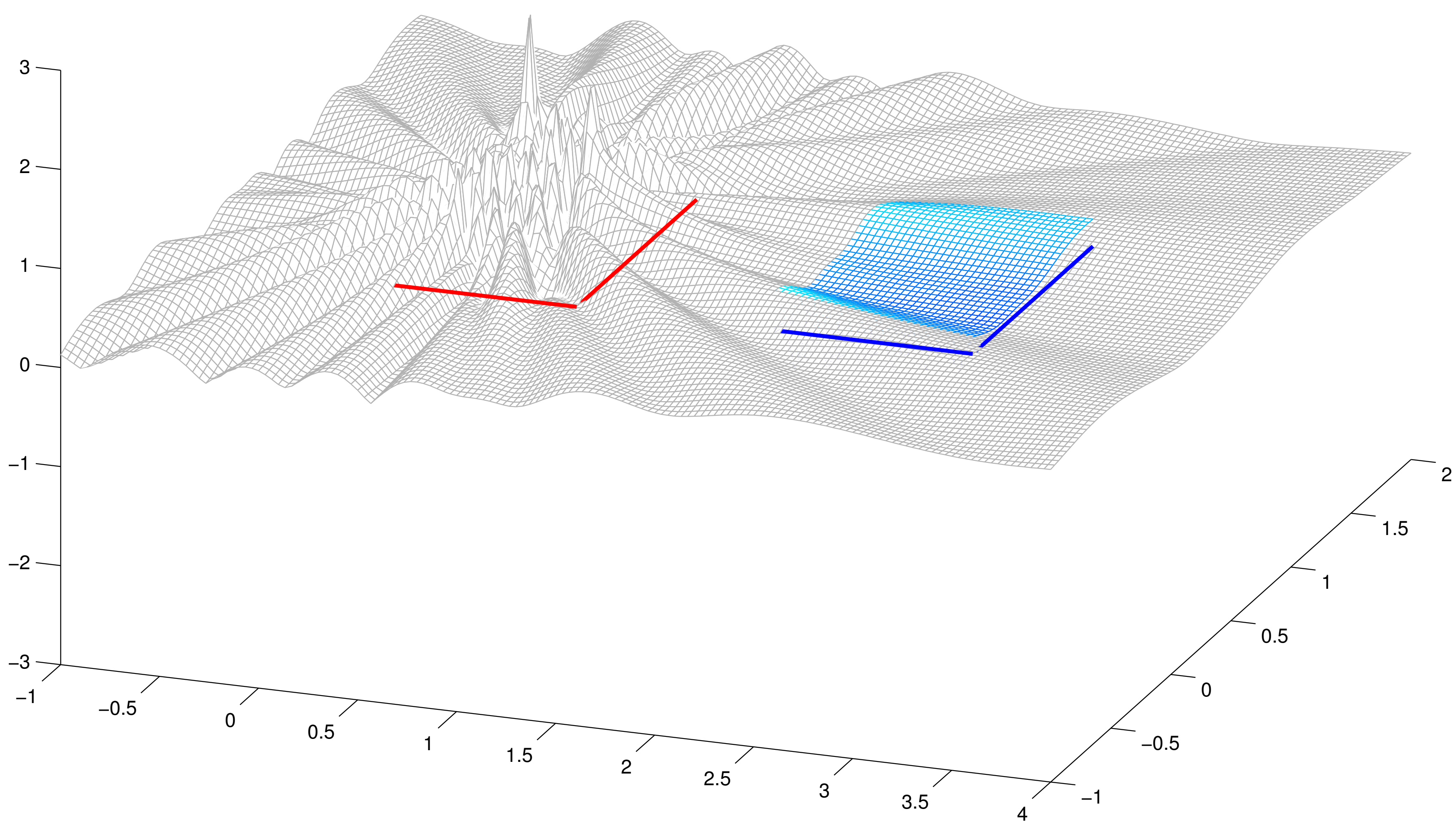
Helmholtz problem. Side of boxes = 6.37 lambda

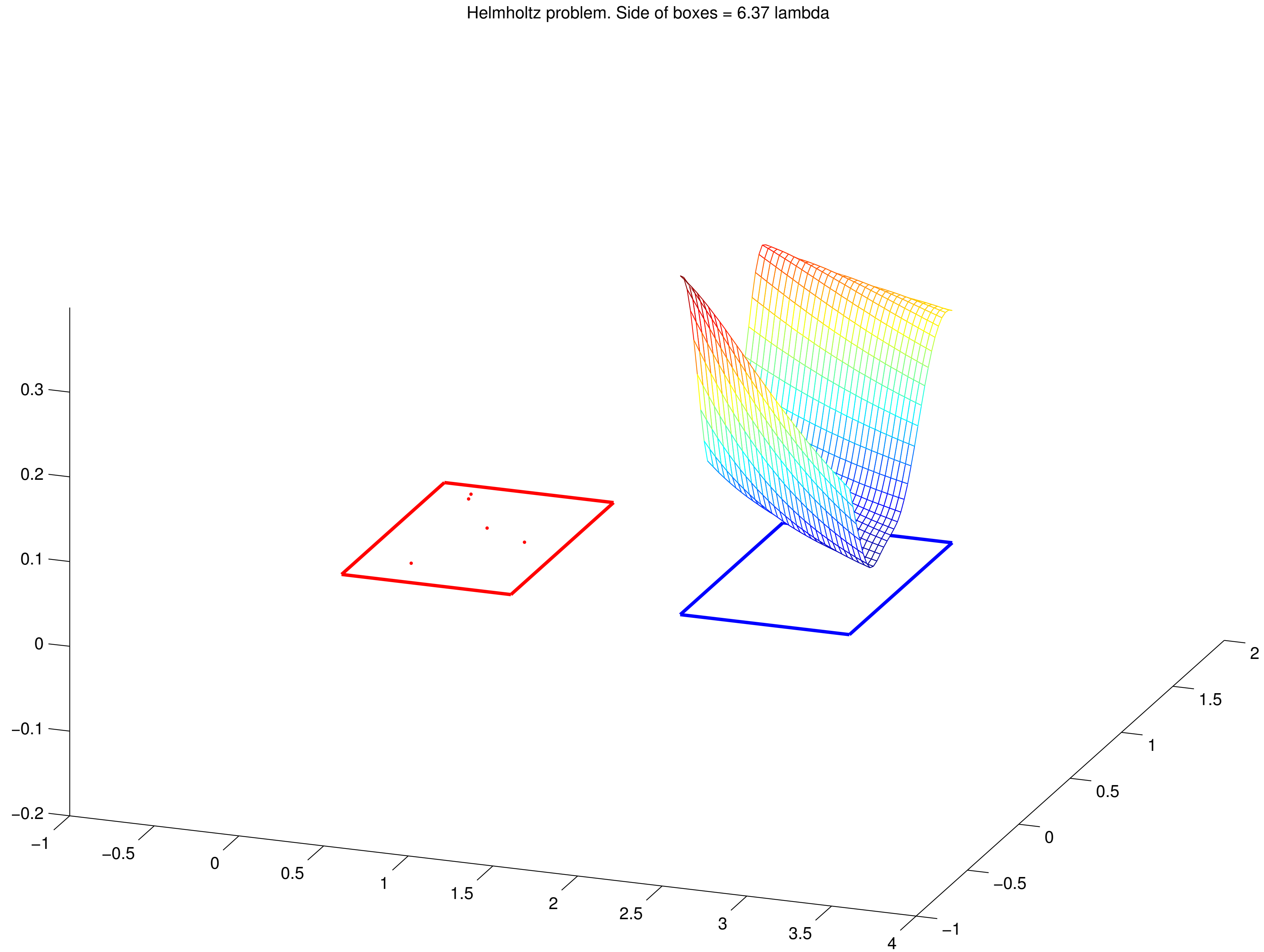


Absolute value of field generated by the sources (truncated — the peaks go to infinity).

Example of solution of the Helmholtz equation $-\Delta u - \kappa^2 u = g$

Suppose we are given point charges $\{q_j\}_{j=1}^5$ in a "source domain" Ω_s.

We are interested in the potential in a "target domain" Ω_t. *Now for larger κ!*



Absolute value of field generated by the sources (truncated — the peaks go to infinity).

Superficially, almost everything we've discussed for the Laplace case carries right over to the Helmholtz case.

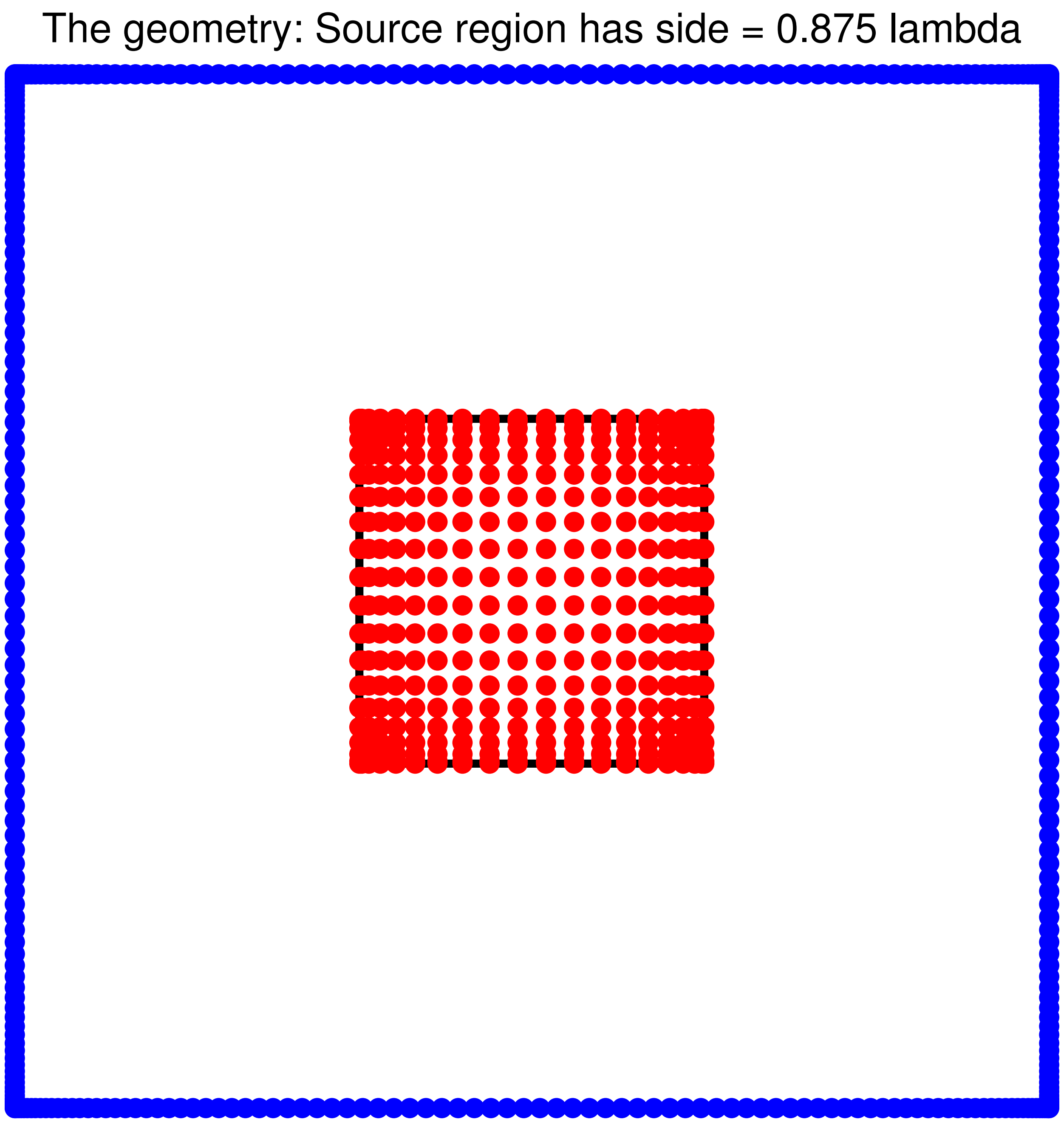
For instance, there is a "multipole expansion." Set

$$S_n(\boldsymbol{x}) = H_n^{(1)}(\kappa r)\, e^{-in\theta}$$
$$R_n(\boldsymbol{x}) = J_n(\kappa r)\, e^{in\theta}.$$

Then

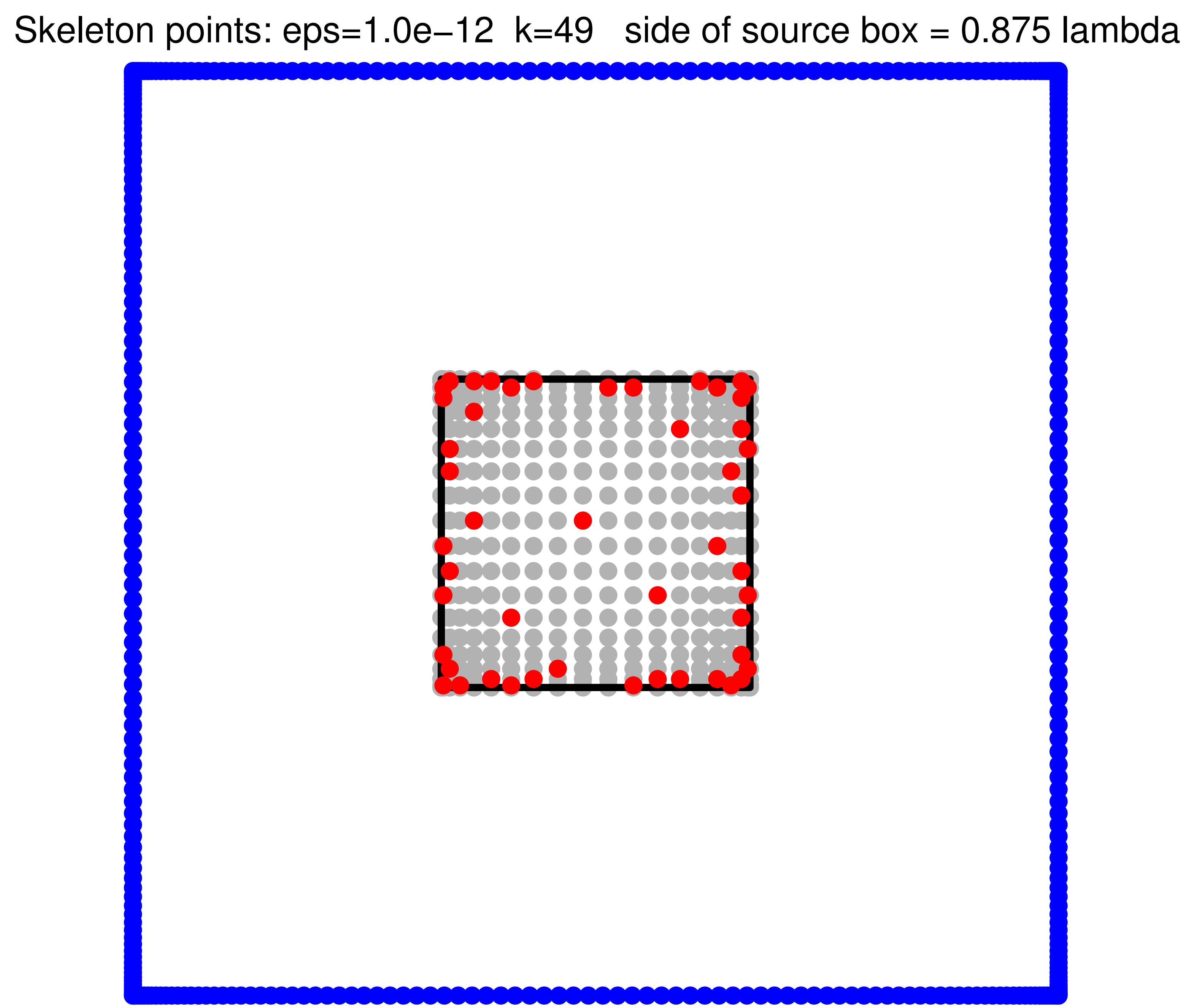
$$H_0^{(1)}(\kappa|\boldsymbol{x} - \boldsymbol{y}|) = \sum_{n=-\infty}^{\infty} S_n(\boldsymbol{x})\, R_n(\boldsymbol{y}), \qquad \text{when } |\boldsymbol{x}| > |\boldsymbol{y}|.$$

Example: Two squares — Helmholtz — small wave number.



Sources in a box of side length 0.9λ*, targets on a box of side length* 2.6λ*.*

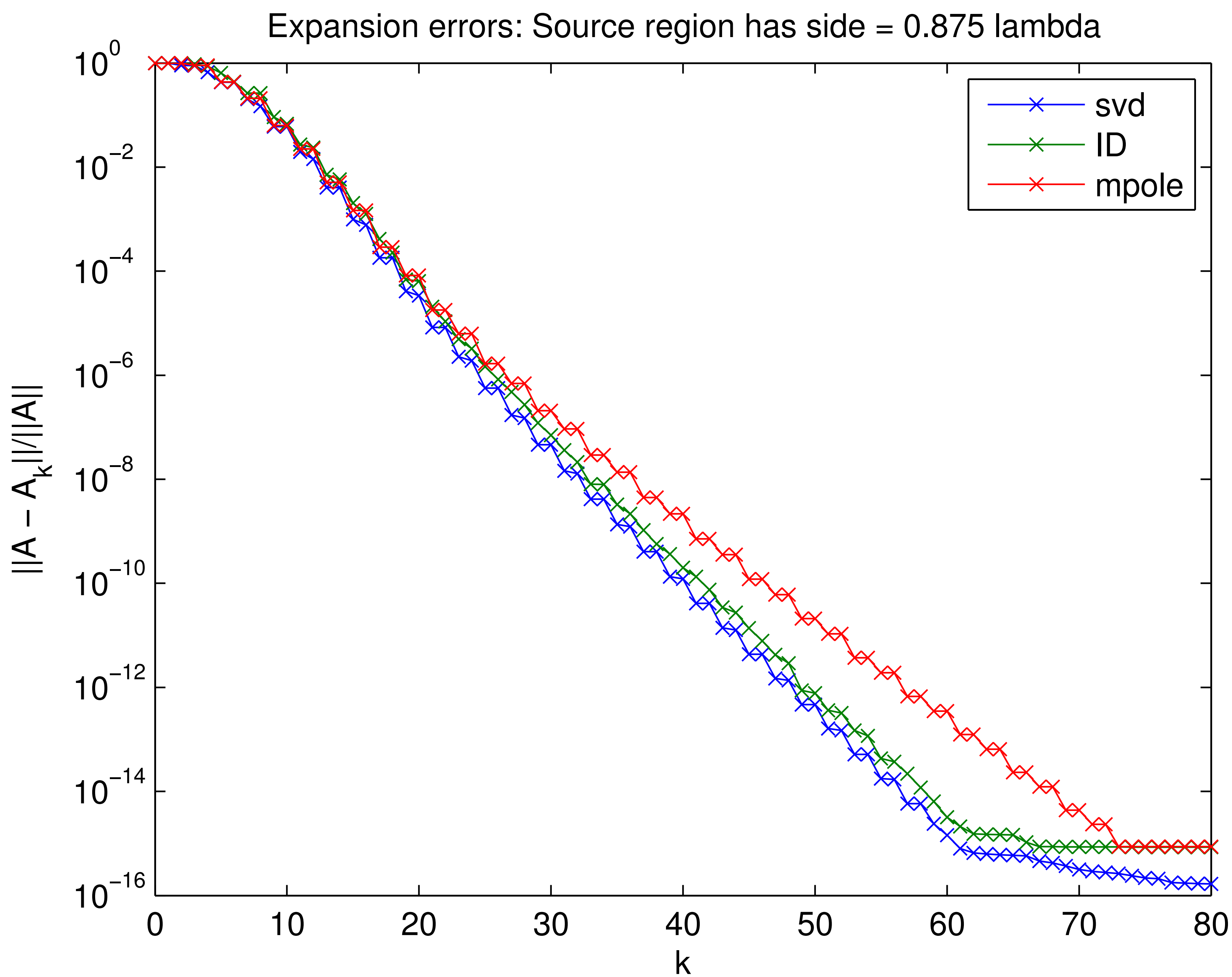
Example: Two squares — Helmholtz — small wave number.



Sources in a box of side length 0.9λ*, targets on a box of side length* 2.6λ*.*

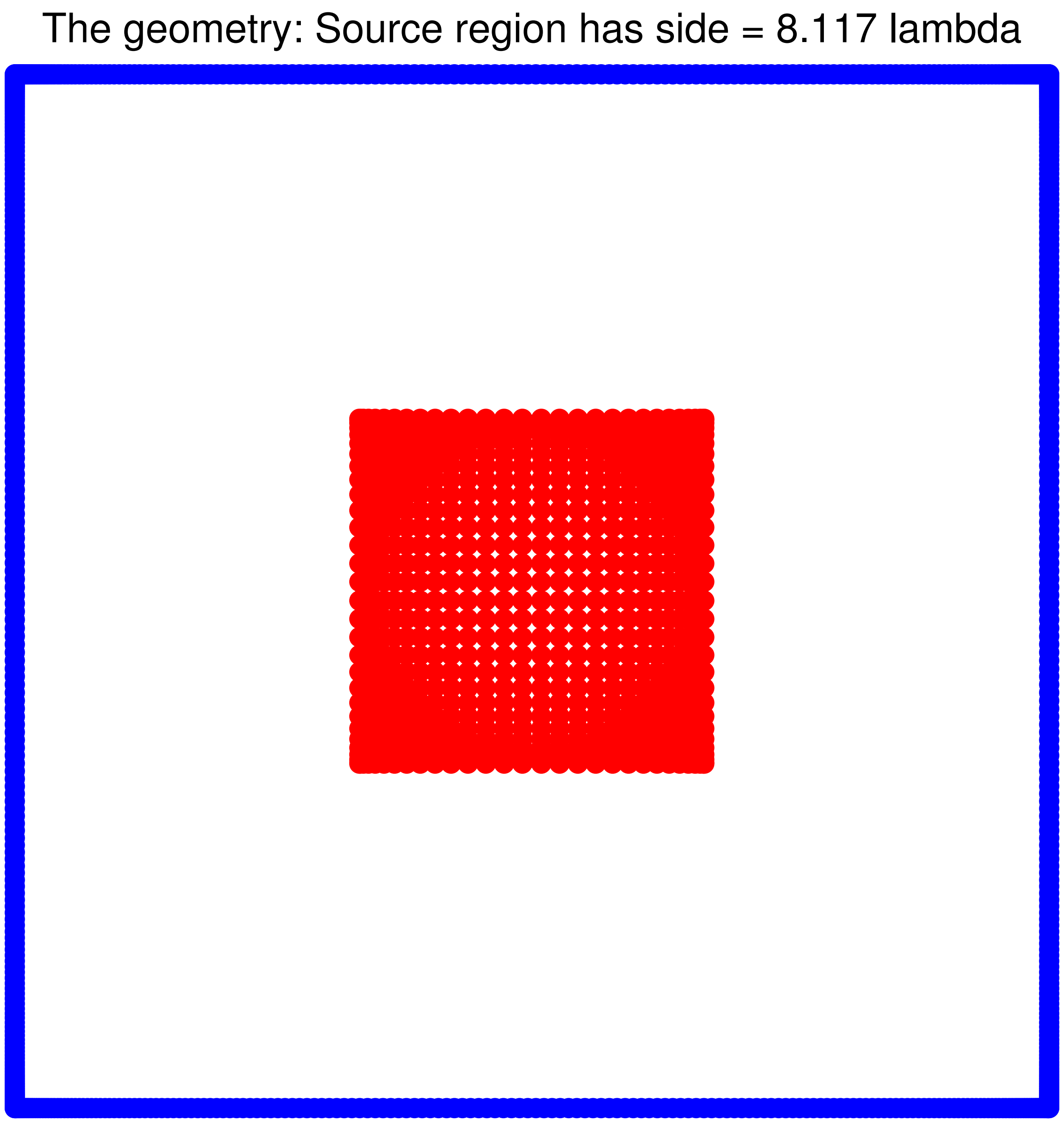
Skeleton to precision $\varepsilon = 10^{-12}$*, which requires* $k = 49$*.*

Example: Two squares — Helmholtz — small wave number.



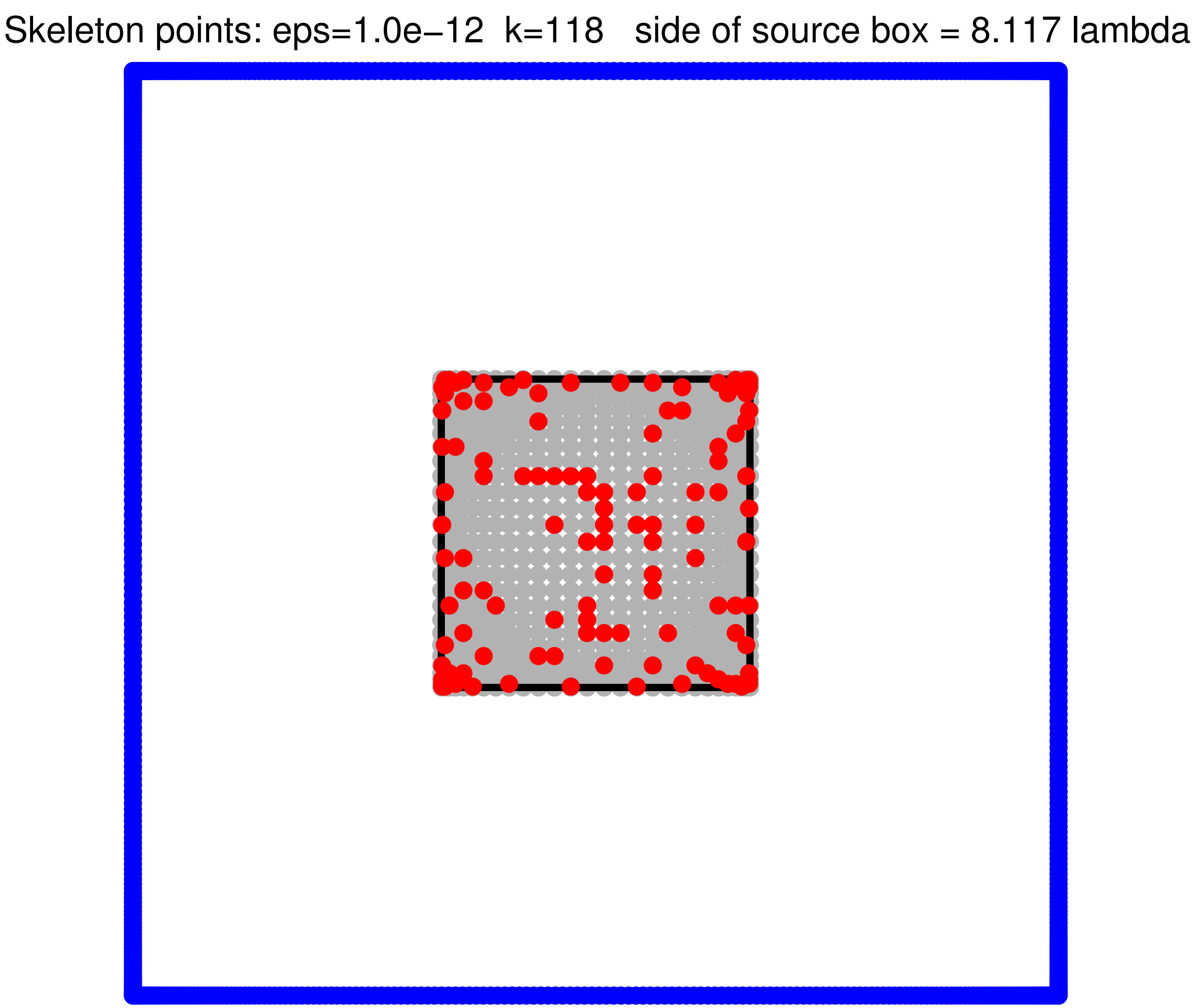
Sources in a box of side length 0.9λ*, targets on a box of side length* 2.6λ*.*

Example: Two squares — Helmholtz — medium wave number.



Sources in a box of side length 8.1λ*, targets on a box of side length* 24.4λ*.*

Example: Two squares — Helmholtz — medium wave number.

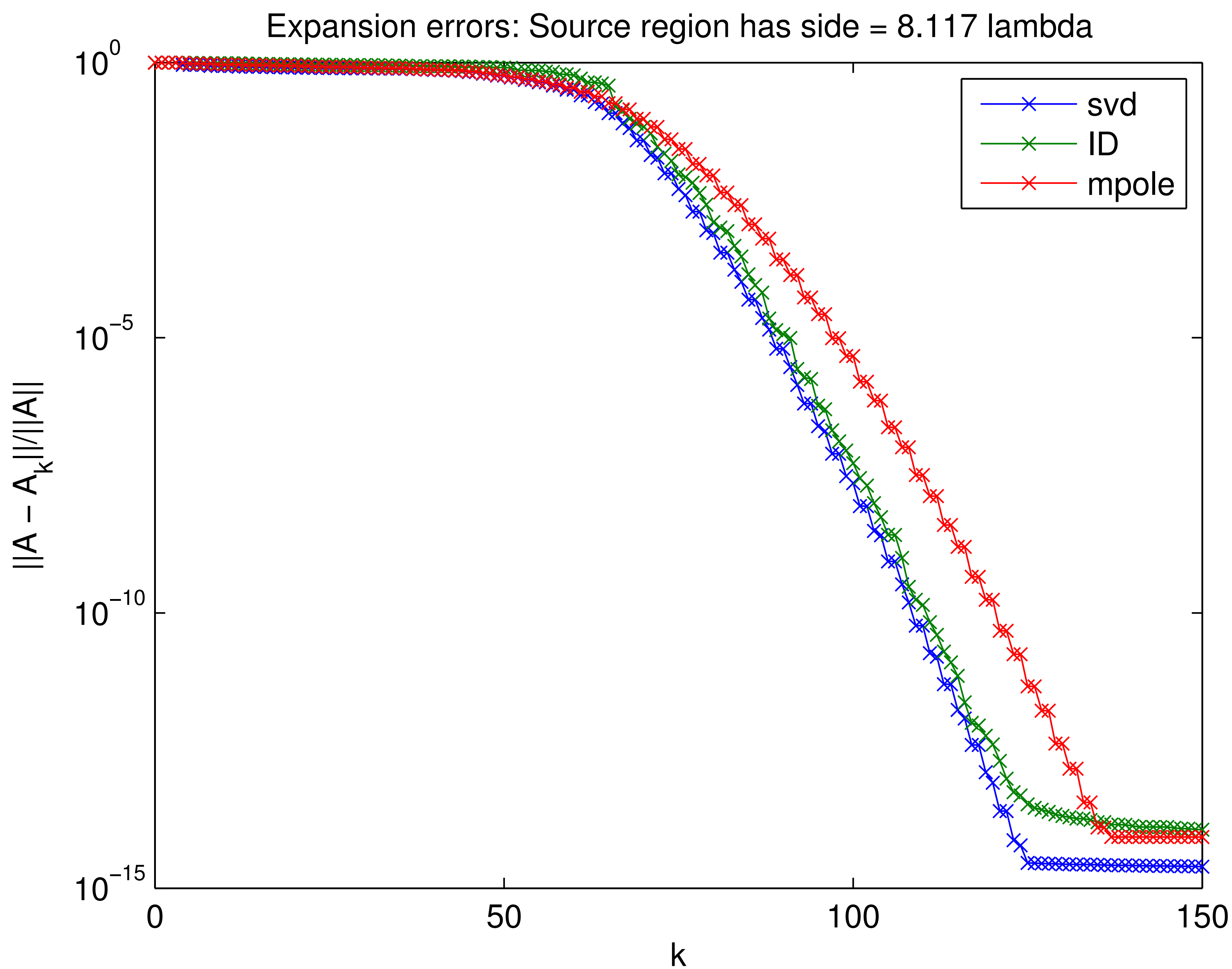


Sources in a box of side length 8.1λ*, targets on a box of side length* 24.4λ*.*

Skeleton to precision $\varepsilon = 10^{-12}$*, which requires* $k = 118$*.*

Observe how many points are now internal — they used to cluster along the boundary.

Example: Two squares — Helmholtz — medium wave number.



Sources in a box of side length 8.1λ*, targets on a box of side length* 24.4λ*.*

Complications with the Helmholtz problem:

1. Decay of singular values starts happening only for *sub-wave-length scales.*

   For geometries that are "large" in terms of wave-lengths, rank considerations alone will be not be sufficient.

2. Resonances are possible. Consider for instance the Dirichlet boundary value problem:

$$\begin{cases} -\Delta u(\boldsymbol{x}) - \kappa^2 u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega, \end{cases}$$

   where Ω is a "simple" finite domain. There exist a sequence of wave-numbers $0 \leq \kappa_1 \leq \kappa_2 \leq \kappa_3 \leq \cdots$ for which the BVP is ill-posed. These are the numbers for which κ_j^2 is an eigenvalue of $-\Delta$. At these "resonant wave-numbers" there exist non-trivial solutions for $f = 0$.

   This creates complications in setting up proxy charges (need *two* layers, or use both monopoles and dipoles, e.g.).

3. While the Laplace equation has a simple "maximum principle" (a harmonic function attains its max on the boundary), the Helmholtz equation is more complicated.

4. Etc.

Similar schemes have been proposed by many researchers:

1993 - C.R. Anderson

1995 - C.L. Berman

1996 - E. Michielssen, A. Boag

1999 - J. Makino

2004 - L. Ying, G. Biros, D. Zorin

A mathematical foundation:

1996 - M. Gu, S. Eisenstat