A Class of Stable Multi-State Time-Reversible Cellular Automata with Rich Particle Content

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Abstract

A new class of stable multi-state Cellular Automata (CA) is derived. They serve to significantly extend the two-state, time-irreversible CA proposed by Park, Steiglitz, and Thurston. These new CA are time-reversible dynamical systems possessing an enormous array of coherent particle-like structures. The particle interaction picture is surprisingly rich and includes particle production as well.

In recent years there has been a significant interest in the various properties and applications of Cellular Automata (CA). From a general point of view CA can be described as a dynamical system in discrete space and time whose field variables take only finitely many values (say 0 and 1, or values in a finite field).

Indeed von Neumann [1] recognized many years ago the importance CA could play in the theory of computation, and Conway [2] in his famous "game of life" automata found an intriguing system capable of supporting a limited number of coherent structures.

Many workers now believe that CA, due to its simplicity and remarkable properties, may play a rather important (if not fundamental) role in describing nature (c.f. [3] for

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review). Important applications include such fields as fluid dynamics, chemical and biological modeling, etc. [4, 5], and there already is a large literature discussing numerous computer simulations of stochastic and intrinsic properties of CA (see also [6] for a recent conference proceedings). However, many of these properties were not studied theoretically due to the extreme nonlinearity of the models.

In [7, 8], Park, Steiglitz, and Thurston proposed a remarkable CA, called the Parity Rule Filter Automata (PRFA), which exhibits a number of important properties. These include a wide range of particle-like structures, a subclass of particles with soliton-like behavior, etc. The PRFA has been studied extensively by a number of authors [9] - [12]. However, the lack of time-reversibility in the PRFA suggests that there may yet be a more fundamental underlying CA, sharing many of its desirable features.

In the present letter we will show that this is indeed the case. We will derive a new class of CA which significantly extends the PRFA, is time-reversible, possesses an enormous particle content (including all particles in the PRFA itself), and exhibits rich interaction phenomena including particle production.

In order to make this letter self-contained, we begin by summarizing the definitions and the basic results for the PRFA. This is a 1+1 dynamical system with discrete space-time $\mathbb{Z} \times \mathbb{Z}$ (here $\mathbb{Z}$ denotes the set of all integers) with a field variable $x_i^t \in \{0,1\}$, $i, t \in \mathbb{Z}$. It is assumed that the initial data, $C^0 = \{x_i^0\}_{i \in \mathbb{Z}}$ at $t = 0$ is localized (i.e. C^0 contains finitely many 1's). Then the time-evolution (i.e. the equations of motion) is given by the following parity rule

$$x_i^{t+1} = \begin{cases} 1 & \text{if } S_{i,2}^t \text{ is even and nonzero} \\ 0 & \text{if } S_{i,2}^t \text{ is odd or zero} \end{cases} \tag{1}$$

where $S_{i,2}^t = \sum_{j=1}^r x_{i-j}^{t+1} + \sum_{j=0}^r x_{i+j}^t$ and the integer parameter $r \geq 1$, called the radius, describes the range of interaction. This rule is clearly implicit and requires sweeping from the left ($x_i^t = 0$ for all $i < N$ for some $N \in \mathbb{Z}$) which is consistent with localized initial data.

An equivalent explicit formulation of the PRFA, the so-called Fast Rule Theorem (FRT), was given in [9], and consists of the following. First, for every t define a subset $B(t) \subset \mathbb{Z}$ by the following inductive procedure:

(i) Sweeping from the left, place the index (spatial coordinate) of the first non-zero site among $C^t = \{x_i^t\}_{i \in \mathbb{Z}}$ in $B(t)$;

(ii) In increments of $r+1$, place $i+r+1$ in $B(t)$, if there is at least one non-zero x_i^t in the r sites to the right of the current index in $B(t)$;

(iii) If there are r zero consecutive values of x_i^t to the right of the current index in $B(t)$, go on to the next non-zero x_i^t, place the corresponding index in $B(t)$, and repeat steps **(ii)**-**(iii)**.

The time-evolution of the automata according to the FRT is then given by

$$x_{i-r}^{t+1} = \begin{cases} x_i^t & \text{if } i \notin B(t) \\ 1 - x_i^t & \text{if } i \in B(t). \end{cases} \tag{2}$$

The FRT can be used effectively to demonstrate that the PRFA is a particularly rich but yet simple dynamical system. The field variables are then $x_i^t \in \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$; i.e. a finite field of two elements, 0 and 1. It has the following properties (see [9] - [12]).

I Stability. For every t the set $B(t)$ is finite and therefore at every time step there are finitely many 1's. Moreover, as t increases the position of the right-most 1 does not move to the right so that the field configuration $C^t = \{x_i^t\}_{i\in\mathbb{Z}}$ moves to the left. Observe that by Eq. (2), a galilean shift is always available.

II Time-Irreversibility. As a dynamical system the PRFA contains dissipation, i.e. is non-conservative, and is therefore time-irreversible. The simplest initial data, a single 1 in an infinity of zeros, evolves to the zero configuration in one time step. This state is called a prenull. More generally, if the configuration C^t contains prenulls, i.e. there exists (at least one) $i \in B(t)$ such that $x_i^t = 1$ and $x_{i+j}^t = 0, j = 1, 2, \ldots r$, then the one step evolution $C^t \to C^{t+1}$ is irreversible; i.e. the state C^t cannot be uniquely determined from C^{t+1} by "going backwards".

Denote by $\mathcal{C}_r$ the (reversible) subset of set $\mathcal{C}$ of all initial data consisting of those states in which there appears no prenull in evolution forward and backward in time, and let $\mathcal{C}_i$ (irreversible) be its complement in $\mathcal{C}$. Then it follows from the FRT that the restriction of the PRFA to $\mathcal{C}_r$ is time-reversible whereas its restriction to $\mathcal{C}_i$ is irreversible. The irreversibility is responsible for the complications discussed in part $\mathbf{IV}_2$, below.

III Particle Content. The time-evolution of PRFA exhibits many different coherent structures. Each is represented by initial data $C^0 = \{x_i^0\}_{i\in\mathbb{Z}}$ which evolves in a time-reversible way as a freely moving localized object. In other words there exist integers p, called the period, and d, called the displacement, such that for all $i \in \mathbf{Z}$ $x_{i-d}^{t+p} = x_i^t$. Such initial data are naturally associated with particles. The time-evolution of a particle is translation to the left by d units after p successive time steps. Its velocity is defined as $v = \frac{d}{p}$, where $0 \leq v \leq r - 1$. When $p = 1$ the particle moves like a point-like object; a localized wave retaining its "shape". Particles with $p > 1$ have additional internal structure (inner degrees of freedom) which exhibits itself during the period. There exist many particles with various p and d, [10, 12], and a systematic way of constructing them via linear difference equations is given in [12].

Among all particles, one can distinguish basic particles which behave like solitons [11]. These correspond to initial data C^0 localized in the space interval of the length not greater than $r + 1$ units; the zero configuration and prenull are excluded.

IV Interaction and Scattering of Particles. This concept refers to the large time behavior of the initial data consisting of spatially separated particles. The corresponding asymptotic picture is described by different coherent structures–asymptotic states–which depend on the interaction between particles. There are two major cases.

IV$_1$ Solitonic (time-reversible and elastic) scattering. This process describes scattering of basic particles and is similar to the scattering in 1+1 integrable solitonic dynamical systems namely, the initial state C^0 consisting of spatially separated basic particles $A_1, A_2, \ldots, A_n$ ordered by their velocities $v_1 < v_2 < \ldots < v_n$ evolves through a complicated interaction into an asymptotic state C^∞ describing the free motion of the same particles $A_n, \ldots, A_2, A_1$ arranged in the opposite order. (See Fig. 1(a) for a typical two-particle soliton scattering). Additionally there is a set of more structurally complicated particles which also interact solitonically with each other (e.g. $p = 1$ waves).

IV$_2$ Non-Solitonic (time-irreversible) scattering. This phenomena is due to the existence of the set $\mathcal{C}_i$ of time-irreversible initial data (property **II** above). In the process of evolution, the states from $\mathcal{C}_i$ produce "false" particles, containing prenulls, which die in the next time step. (It is the prenull state which is responsible for the non-conservation of the energy-type functional introduced in [10].) There are three typical examples of the non-solitonic scattering:

a) Gluing. Particles A and B interact and form another particle C (see Fig. 1(b)).

b) Inelastic. Scattering of two particles A and B results in two different particles B' and A', neither of which is A or B, (see Fig. 1(c)).

c) Irreversible Solitonic. Scattering of two particles A and B results in the same particles B and A, but this process, due to the existence of prenulls, is not time-reversible, (see Fig. 1(d)).

Thus we have seen that the PRFA possesses many remarkable properties but fundamentally speaking, it lacks time-reversibility.

The aim of this letter is to present a natural modification of the PRFA which results in a new class of time-reversible CA, which is easily extended to a multi-state CA. To achieve this, notice first that the parity rule Eq. (1) is equivalent to the following difference relation in the finite field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$

$$\sum_{j=0}^{r} x_{i-j}^{t+1} \equiv \sum_{j=0}^{r} x_{i+j}^{t} + \delta_2(x_i^t) \prod_{j=1}^{r} \delta_2(x_{i-j}^{t+1}) \delta_2(x_{i+j}^{t}) - 1 \tag{3}$$

Here equality $\equiv$ is understood mod 2, i.e. $(1+1) \bmod 2 = 1+1 \equiv 0$, and $\delta_k(x)$ is a "delta function mod k"; i.e. $\delta_2(x) = 1$ if $x = 0 \bmod 2$ and 0 otherwise. Rewriting Eq. (3) in the form $x_i^{t+1} \equiv S_{i,2}^t + (1 - x_i^t) \prod_{j=1}^{r} \delta_2(x_{i-j}^{t+1}) \delta_2(x_{i+j}^t) - 1$, we can interpret $S_{i,2}^t$ as a linear term and the rest as a highly nonlinear perturbation.

We note that Eq. (3) admits natural multi-state generalizations. Consider the finite ring $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$ consisting of q elements $0, 1, \ldots, q-1$ ($\mathbb{F}_q$ is a field if q is prime). Now assume that the field variable $x_i^t \in \mathbb{F}_q$ and define its time evolution by

$$\sum_{j=0}^{r} x_{i-j}^{t+1} \equiv \sum_{j=0}^{r} x_{i+j}^{t} + \delta_q(x_i^t) \prod_{j=1}^{r} \delta_q(x_{i-j}^{t+1}) \delta_q(x_{i+j}^{t}) - 1 \tag{4}$$

where now equality $\equiv$ is understood mod q, i.e. $1 + q - 1 \equiv 0$. Eq. (4) is equivalent to the following q-state parity rule

$$x_i^{t+1} = \begin{cases} (S_{i,q}^t - 1) \bmod q & \text{if } S_{i,q}^t \neq 0 \\ 0 & \text{if } S_{i,q}^t = 0, \end{cases} \tag{5}$$

where now $S_{i,q}^t = (q-1)\sum_{j=1}^r x_{i-j}^{t+1} + \sum_{j=0}^r x_{i+j}^t$.

In turn, Eq. (5) is equivalent to the following q-state Fast Rule Theorem

$$x_{i-r}^{t+1} = \begin{cases} x_i^t & \text{if } i \notin B(t) \\ (x_i^t - 1) \bmod q & \text{if } i \in B(t), \end{cases} \tag{6}$$

where the set $B(t)$ was defined earlier by **(i)-(iii)**.

This multi-state CA has all of the properties **I-IV** of the PRFA (with yet a wider particle content); but as with the PRFA it is time-irreversible; i.e. a prenull vanishes at the next time step.

An inspection of Eq. (3) shows that the reason for dissipation and time-irreversibility is due to the presence of the factor $\delta_2(x_i^t)$ in the product, which violates the symmetry $x_{i+j}^t \leftrightarrow x_{i-j}^{t+1}$. Thus we will define a new two-state CA by the following difference relation

$$\sum_{j=0}^r x_{i-j}^{t+1} \equiv \sum_{j=0}^r x_{i+j}^t + \prod_{j=1}^r \delta_2(x_{i-j}^{t+1})\delta_2(x_{i+j}^t) - 1 \tag{7}$$

which has the symmetry $x_{i+j}^t \leftrightarrow x_{i-j}^{t+1}$, and therefore is time-reversible. Backward evolution in time, from C^{t+1} to C^t is given by the same Eq. (7) which in order to solve requires sweeping from the right. Equivalently, the time-evolution in this new CA can be described by the reversible parity rule

$$x_i^{t+1} = \begin{cases} 1 & \text{if } S_{i,2}^t \text{ is even and non-zero or } x_i^t = S_{i,2}^t = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

and alternatively by the Reversible Fast Rule Theorem (RFRT), given exactly by Eq. (2) where the set $B(t)$ should be replaced by the set $\tilde{B}(t)$ defined as follows: steps (i) and (ii) in the definition of $B(t)$ remain unchanged, whereas step (iii) is replaced by the following

(iii)$'$ If $i \in \tilde{B}(t)$ and $x_{i+j}^t = 0, j = 1, \ldots r$ then either $i + r \in \tilde{B}(t)$ if $x_i^t = 1$ or, if $x_i^t = 0$, the index of the next non-zero x_i^t in C^t belongs to $\tilde{B}(t)$.

In particular, the RFRT shows that the new CA is stable. From Eqs. (7),(8) it follows that restriction of the new CA to the subset $\mathcal{C}_r$ of initial data coincides with the PRFA, therefore it possesses all the desirable properties **I-IV**$_1$! Moreover, due to time-reversibility it "cures" the bad property **IV**$_2$ of the PRFA. Namely, the time evolution according to Eqs. (7),(8) shows that in this new CA the prenulls do not vanish but instead are now elementary basic particle solutions stationary in time. They represent the simplest of all particles, which we call prebasic particles. With these particles, the reversible rule possesses an infinite collection of particles, with highly non-trivial interaction properties.

For instance, even in the simplest case $r = 1$ (which is totally trivial for the PRFA), appropriate initial data produce particles with zero velocity and arbitrarily large periods exhibiting interesting and complicated internal motion. In the case $r \geq 2$, the situation is more dramatic: in particular, initial data consisting of certain sequences of consecutive 1's (with lengths belonging to certain infinite arithmetic progressions depending on r) give rise to particles with arbitrarily large periods.

In addition to the solitonic interactions covered by case $\mathbf{IV}_1$, compare Figs. 1(a) and 2(a), the new CA exhibits highly non-trivial particle interactions. Observe that the bound state shown in Fig. 1(b) now corresponds to *particle production*, see Fig. 2(b). This example illustrates that scattering of particles A and B results in a new particle C and a number of emitted prebasic particles. Reversing the time order implies that particle C bombarded by a beam of prebasic particles splits into two particles A and B. Needless to say, this picture is very suggestive: prebasic particles playing various roles; in particular being responsible for particle production. In a sense, this new CA might serve as a "toy" model of fundamental physics. Comparing the data shown in Fig. 1(c) with that shown in Fig. 2(c), we see that non-solitonic interactions which do not produce prebasic particles are indeed possible.

The initial condition shown in Fig. 1(d) consists of two particles $A = 101101101$ and $B = 1111010111$ separated by 14 zeros. Under the reversible rule this data exhibits remarkable behavior. After 10000 time steps these data have finally separated into a number of nontrivial particles consisting of # isolated particles, 2 pairs of bound states (such as those in Fig. 1(b)), and 9 prebasic particles. This behavior is entirely unexpected and graphically illustrates the huge complexity of this new rule.

Finally, we present a multi-state generalization of the proposed CA; the set $\mathbb{F}_q = \{0, 1, \ldots, q-1\}$ of values x_i^t might be considered as higher spins ("color") or perhaps other internal degrees of freedom. We define a new difference relation (compare with Eq. (4)) as follows:

$$\sum_{j=0}^{r} x_{i-j}^{t+1} \equiv \sum_{j=0}^{r} x_{i+j}^{t} + \left[\delta_q(x_i^t) - \delta_q(x_i^t - 1)\right] \prod_{j=1}^{r} \delta_q(x_{i-j}^{t+1}) \delta_q(x_{i+j}^{t}) - 1 \tag{9}$$

where $x_i^t \in \mathbb{F}_q$ for all $i, t \in \mathbb{Z}$. The corresponding q-state RFRT takes the form

$$x_{i-r}^{t+1} = \begin{cases} x_i^t & \text{if } i \notin \tilde{B}(t) \\ (x_i^t - 1) \bmod q & \text{if } i \in \tilde{B}(t), \end{cases} \tag{10}$$

where the set $\tilde{B}(t)$ was defined earlier by **(i)**-**(iii)**$'$. Eqs. (9) and (10) provide natural multi-state generalizations of the two-state time-reversible CA Eq. (7). Qualitatively, it has the same properties as a two-state CA (with even richer particle content and interactions due to the "color"). It can be visualized as a spatial array of "dials" with q positions. Time-evolution of each state consists of rotating some of the dials one step counter-clockwise according to the set $\tilde{B}(t)$, say, while running time backwards means rotating the dials one step clockwise.

The FRT for the time-evolution backwards is given by

$$x_{i+r}^{t} = \begin{cases} x_i^{t+1} & \text{if } i \notin \mathcal{B}(t) \\ (x_i^{t+1} + 1) \bmod q & \text{if } i \in \mathcal{B}(t), \end{cases} \tag{11}$$

where the set $\mathcal{B}(t)$ is defined in the same manner as $\tilde{B}(t)$ but with the sweeping going from right to left.

One can consider another possible multi-state generalization of the proposed CA. It consists of modifying the difference relation (9) by dropping the coefficient $\delta_q(x_i^t) - \delta_q(x_i^t - 1)$ of the product term. Though this CA obviously has the symmetry $x_{i+j}^t \leftrightarrow x_{i-j}^{t+1}$, certain initial data (e.g. a single $x \in \mathbb{F}_2, x \neq 0, 1$) blows up in finite time so that this obvious generalization leads to an unstable CA.

Thus the novel class of proposed time-reversible multi-state CA's exhibits intriguing regularity and a large variety of coherent structures. Further investigation by computer simulation and by analytical techniques is highly desirable, and will certainly lead to a better understanding of this fascinating phenomena. Given this vast array of coherent particle-like solutions, we believe that this new CA, and models like this one, may find valuable applications in many areas of science (e.g. nonlinear physics, information and computation theory, etc.) These models may be viewed as the most basic integrable (i.e. an arbitrarily large number of solutions may be constructed analytically) time-reversible dynamical system over a finite field $\mathbb{F}_q$. In 1+1 integrable solitonic systems, the infinite number of conservation laws and properties of the system prohibit particle production. However over a finite field, these properties are clearly less restrictive allowing the CA to be consistent with non-elastic particle interaction.

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Fig. 1. Particle interactions defined by Eq. (1). (a) solitonic reversible; (b) bound state irreversible; (c) non-solitonic irreversible; (d) solitonic reversible.

Fig. 2. Particle interactions defined by Eq. (7). (a) solitonic; (b)-(d) non-solitonic. All interactions are reversible.