Soliton Cellular Automata

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March 28, 1994

Introduction

Cellular Automata (CA) have been applied to physical systems ranging from biology and chemistry, to physics and the engineering sciences, most notably fluid dynamics. One of the most famous CA is LIFE:

- Life was invented in 1968 by Cambridge mathematician J.H. Conway.
- Martin Gardner popularized the game in *Scientific American* in the early 1970's.
- Life is played on an infinite 2-dimensional grid (checkerboard).
- Each cell has eight neighbors (von Neumann neighborhood).
- Each cell is in 1 of 2 states:

$$0 \equiv Dead \qquad\qquad 1 \equiv Alive$$

- At time $t+1$ the state of a cell depends on the number of living cells in its neighborhood at the previous time t.
- All cells are updated simultaneously according to the following rules:

  (i) If a dead cell has precisely 3 living neighbors at time t, then that cell is "born" at time $t+1$, otherwise it remains dead.

  (ii) If a living cell has 2 or 3 neighbors at time t, then it remains living at time $t+1$.

  (iii) If a living cell has ≤ 1 or ≥ 4 living neighbors at time t, then it dies at time $t+1$ from "loneliness" or "overpopulation", respectively.

CA vs FA

Automata are partitioned into two broad categories depending on the type of previous information used to update the current state.

CA's evolve according to a rule using one previous time step to determine the state of a given cell, x_i^t. This rule is of the form

$$x_i^{t+1} = F(r, x_i^t) \tag{1}$$

where F is called the Global Update Rule (GUP), and r defines the neighborhood influencing the cell x_i^t.

We are interested in a class of one dimensional automata called Filter Automata (FA) which have rules of the form

$$x_i^{t+1} = F(r, x_i^t, x_i^{t+1}) \tag{2}$$

Note that both previous, x_i^t, and current, x_i^{t+1}, information is used to determine x_i^{t+1}.

The particular GUP associated with (2) was introduced by Steiglita, Kamal, and Watson of Princeton University, and is called the Parity Rule Filter Automata (PRFA). The PRFA is given by the following

$$x_i^{t+1} = \begin{cases} 1 & \text{if } S(x_i^{t+1}) \text{ is even and nonzero} \\ 0 & \text{if } S(x_i^{t+1}) \text{ is odd or zero} \end{cases} \tag{3}$$

where

$$S(x_i^{t+1}) = \sum_{j=1}^{r} x_{i-j}^{t+1} + \sum_{j=0}^{r} x_{i+j}^{t} \tag{4}$$

Figure 2 shows a random IC evolved for a number of time steps. We see that a number of stable patterns, called "particles", have emerged. The PRFA supports a number of these particles. Figure 3 shows what we call a solitonic interaction, illustrating the reason for our interest in the PRFA.

FRT

The PRFA as given by (3) and (4) is analytically difficult to manipulate, prompting the development of the Fast Rule Theorem (FRT) by Papatheodorou, Ablowitz, and Saridakis.

The FRT is given by

$$x_{i-r}^{t+1} = \begin{cases} x_i^t & \text{if } i \notin B(t) \\ \overline{x}_i^t & \text{if } i \in B(t) \end{cases} \tag{5}$$

where $\overline{a} = (a+1) \bmod 2$, and is the complement of a. $B(t)$ is called the set of box sites at time t, and contains space indices constructed as follows:

1. Place the index of the first 1 in $B(t)$
2. In increments of $r+1$, place the corresponding index in $B(t)$, unless there exist at least $r+1$ zeros after a previously included index, in this case go on to the next 1.

A number of general results have been obtained using the FRT, proving its usefulness as an analytical tool. Much research has been directed at particles called "basic strings", which consist of groups of $r+1$ sites, and at period 1 particles, finite sequences of 0's and 1's which repeat themselves at every time step.

A Difference Equation for the PRFA

The PRFA has been written in terms of a difference scheme of the form

$$a_i^{t+1} = \sum_{k=2}^{n} (-1)^k \beta_k \sum_{j_1,\ldots,j_k} a_{j_1} a_{j_2} \ldots a_{j_k} \tag{6}$$

where $n = 2r+1$, $\beta_k = 2^{k-1} - 1$, and $\sum_{j_1,\ldots,j_k}$ denotes the sum over all possible combinations of the products of k different terms.

Difference Equation for Periodic Particles

A periodic particle is specified by a number of parameters:

1. Radius, $r \geq 2$
2. Period, $p \geq 1$
3. p intermediate displacements, d_j for $j = 1, 2, \ldots p$, $0 \leq d_j \leq r - 1$.

We introduce the following functions for convenience

$$\delta_j = \delta(\frac{i - \rho_j}{r+1}) \qquad j = 0, 2, \ldots, p-1 \tag{7}$$

where the δ-function is defined by

$$\delta(k) = \begin{cases} 1 & \text{if } k \in \mathcal{Z}^+ \text{ or } 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

and the constants ρ_j are given iteratively by

$$\begin{aligned} \rho_0 &\equiv 0 \\ \rho_j &= \rho_{j-1} + (r - d_j) \qquad j = 1, 2, \ldots, p \end{aligned} \tag{9}$$

Evaluating Equation (9) at $j = p$, we define

$$\rho = \rho_p = p * r - \sum_{j=1}^{p} d_j \tag{10}$$

It has been shown that the set of difference equations

$$\begin{aligned} x_i &= \delta(\frac{i - \rho_0}{r+1}) \qquad 0 \leq i < \rho \\ x_i &= x_{i-\rho} + (1 - 2x_{i-\rho})\delta(\frac{i - \rho_0}{r+1}) \qquad i \geq \rho \end{aligned} \tag{11}$$

produces spatially periodic sequences of 0's and 1's of the form

$$\mathbf{POPOPO}\ldots.$$

where **P** is non-trivial and begins and ends with a 1, and each **O** contains only 0's. Each **P** was then shown to be a temporally periodic particle of period 1 when evolved by the FRT.

Also, it was shown that period 2 particles can be produced by an analogous set of difference equations

$$\begin{aligned} x_i &= \delta(\frac{i-\rho_0}{r+1}) + \delta(\frac{i-\rho_1}{r+1}) - 2\delta(\frac{i-\rho_0}{r+1})\delta(\frac{i-\rho_1}{r+1}) \qquad 0 \leq i < \rho \qquad (12) \\ x_i &= x_{i-\rho} + (1 - 2x_{i-\rho}) \\ &\quad *(\delta(\frac{i-\rho_0}{r+1}) + \delta(\frac{i-\rho_1}{r+1}) - 2\delta(\frac{i-\rho_0}{r+1})\delta(\frac{i-\rho_1}{r+1})) \qquad i \geq \rho \end{aligned}$$

We have proposed that the general period p difference equation be written as

$$\begin{aligned} x_i &= g_i \qquad\qquad 0 \leq i < \rho \qquad (13) \\ x_i &= x_{i-\rho} f_i + g_i \qquad\qquad i \geq \rho \end{aligned}$$

where

$$f_i = 1 - 2g_i$$

The first equation again defines the ρ initial conditions necessary to evaluate the second equation, resulting in spatially periodic sequences of the above form. The function g_i is, in words, the sum of all combinations of products of individual δ-functions and is given by

$$g_i = a_1 b_1 + a_2 b_2 + \ldots + a_p b_p$$

where

$$a_j = (-1)^{j-1} j \qquad\qquad j = 1, 2, \ldots, p$$

and b_j is the sum of all unique products of j delta functions.

The Future

A number of unanswered questions remain concerning the global behavior of the particles found in the PRFA and the underlying relationship between solutions of, as yet undetermined, PDE's with solitonic solutions. Among these are

- General Evolution Theorem. Figure 2 shows a random initial condition evolving to a finite set of particles ordered by velocity. How can one prove such a general result, that has been observed for all such initial conditions.

- Reversibility of states. The conditions under which a given state can be recreated from future states has been determined, yet the role this has regarding the global behavior of arbitrary initial conditions must be nailed down.

- PDE's. Which, if any, PDE is associated with the discrete structures found in the PRFA?

- Statistical analysis. Is there an appropriate statistical analysis which could be applied to the previous question? Can a rate of organization of random IC's be computed–observe how "quickly" Figure 2 becomes ordered.